

The Search for Liquidity^{*}

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Abstract

Lacking a market for a divisible asset, a seller faces a stream of buyers who arrive at random times with random limit orders. This paper uses search theory to understand how this liquidation sale optimally proceeds. We solve the dynamic programming exercise characterizing his optimal trading behavior. His behavior changes as the asset position falls, reflecting the endogenous time-varying value of the asset position.

Using recursive methods and duality theory, we uncover a new “diminishing returns to optionality” property: The Bellman value function is increasing and concave in the position. The seller therefore takes greater advantage of more generous offers, but his marginal value shifts up as he unwinds his position, making him less willing to trade. Deducing a convex marginal value, we then offer new insights on transactional liquidity in finance for trade size, depth and spreads. We also explore a new dimension of liquidity, namely, the waiting time between trades. Finally, the model is tractable enough to allow for price-quantity bargaining. We find that greater buyer bargaining power is tantamount to greater frictions, and so increases supply, decreases price, and leads to a more liquid induced market.

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1 Introduction

Unable to liquidate long portfolio positions, Lehman-Brothers failed in September 2008. Similar, if less dramatic, liquidation stories were a theme of the Financial Crisis. In the 1990s, the Resolution Trust Corporation (RTC) was tasked with liquidating almost \$400 billion of portfolios of 747 failed savings and loans in the S&L Crisis.¹ Both Lehman-Brothers in 2008 and the RTC in 1990 lacked an organized market to sell their positions, and had to actively search for counterparties under time pressure. This paper develops and solves a dynamic programming model of the time intensive search for trading counterparties for the many situations when there is no formal market or a unique price.

Consider an impatient owner of a large position in a divisible asset who wishes to sell it off. He lacks an organized market and so must slowly sell it off to the stream of buyers who randomly arrive. The arrive according to a Poisson process, each with a random limit order — specifically, a bid price and a purchase cap. Until sale, he possibly earns a flow dividend proportional to his holdings. We solve the dynamic programming exercise characterizing the optimal trading behavior. The seller chooses a supply schedule that optimally solves his recursion. This schedule evolves in the asset position, reflecting the endogenous time-varying option value of having a larger position. In this paper, we characterize the seller’s nonstationary trading behavior as he unwinds his position.

A search for liquidity with indivisible units would entail a 0-1 trading strategy — namely, liquidity today or in the future. But with a divisible asset, the marginal value governs his trading behavior, since it acts as the seller’s marginal cost curve. For whenever a buyer arrives, the seller trades off an immediate sure payoff for an uncertain future payoff — and unlike in standard search theory, this happens *at the margin*.

Our first major result is a new *diminishing returns to optionality* property: The Bellman value function is increasing and concave in the position (Theorems 1–2). As the seller’s asset holdings grow, the share of his Bellman value reflecting search optionality falls. In the limit with a very large position, the seller acts myopically, simply trading according to the present values of flow dividends (Lemma 3). Formally, the value is concave specifically because buyer demands are finite: At smaller positions, the buyers’ purchase caps bind less frequently, and the seller’s opportunity set is accordingly larger. We prove this using recursive methods and convex duality theory of Fenchel.

Diminishing returns to optionality has two key economic implications — first, that

¹“I was on the oversight board of the RTC...Some of the stuff that the RTC wound up with was perfectly liquid and salable. But a big chunk was uncompleted eight-hole golf courses, half-built office towers, and vacant malls. Nobody wanted it. ...Needless to say, the bids were less than 50 percent of the original cost.” — Alan Greenspan (Leonard and Coy, 2012).

the seller takes greater advantage of more generous offers, and second, that his marginal value shifts up as he sells off his position, making him less willing to trade. For the value is lower but the marginal value higher with a smaller position. Our explicit formula for supply (Theorem 3) is separable in the price and position, increasing in price and linearly increasing in position. In particular, supply is inverse to the marginal value, and its slope is thus inverse to that of the marginal value.

We next explore the marginal value of the position. We prove in Lemma 2 that the value function is everywhere differentiable. We cannot use the envelope theorem logic of Benveniste and Scheinkman (1979) since our optimal policy is not necessarily interior. We instead establish the existence of a bounded derivative recursively. Because the marginal value is boundedly positive and boundedly finite, we deduce a *choke price*, below which the seller refuses to trade, and a higher *sell-all price*. Divisibility of the asset position therefore matters for intermediate prices, when the seller optimally partially unwinds his position. The limit marginal value with a very small asset position is the sell-all price, and satisfies the same recursion as the value function does with indivisible assets. We then exploit our recursion for the marginal value to prove that it is not only positive and decreasing, but also convex in the position (Theorem 4). As a result, when positive, the supply is a concave function of the price.

We explore how the value function, trading behavior, and liquidity measures change in the search frictions. We find that the value and marginal value both fall in the search frictions, while the second derivative rises (Theorem 5). Intuitively, in a more frictional environment, the search optionality falls. But since trading opportunities are more rare, the seller trades more in each, and he is less price sensitive. Thus, the value and marginal value functions both become more linear, which we establish recursively.

Our search model offers richer insights on liquidity than typically pursued in finance, since we can offer predictions about both trading behavior and the seller's optimal waiting time between sales. The seller faces a tradeoff between more frequent and predictable trades, and more profitable trades. But as the seller's asset position falls or frictions vanish our transactional liquidity measures worsen, the depth in the induced asset market falls, and the premium over the choke price grows (Theorem 8). On the other hand, waiting times worsen as the asset position falls, and respond ambivalently to search frictions, falling with either a higher meeting rate or interest rate (Theorem 7).

In another advantage of the search environment, the model readily allows price-quantity bargaining, as one might expect in the liquidation process for large trades. In §6, we show that our qualitative results are largely robust to Nash bargaining. In fact, greater bargaining power for buyers is formally equivalent to increased search frictions:

It raises the supply, reduces the negotiated price, and improves the induced market liquidity. All told, the negotiated price and quantity move oppositely, and on balance, the trade value rises for low positions and buyers' reservation values. With bargaining, not only does the supply rise in the position, but the negotiated price falls.

The closest paper to our dynamic programming exercise may well be Lagos and Rocheteau (2009), who develop a search-theoretic equilibrium model of financial intermediation for an over-the-counter market. They assume that heterogeneous investors randomly meet dealers, who trade on their behalf. Assets are divisible, but unlike here, the investor can hold unrestricted nonnegative positions. The asset is traded in an illiquid-competitive inter dealer market, and thus the investor's selling strategy is independent of his asset position — unlike our seller. With a linear utility function such as we assume, they would deduce an all-or-nothing trading strategy. But risk aversion ensures that the investor has an interior solution. A stochastically stationary decision model also underlies the equilibrium model by Lagos, Rocheteau, and Weill (2011).

Other search papers on liquidity assume indivisible assets. Smith (1994) explores how search frictions affect trade with indivisible heterogeneous goods and unit demands where “beauty is in the eye of the beholder”. In Duffie, Garleanu, and Pedersen (2007), agents search for counterparties to trade an indivisible asset, and trades are exogenously fixed by agents' types that follow a Markov chain. Krainer and LeRoy (2002) also explore a search and matching equilibrium model of illiquid indivisible asset valuation.

An instructive counterpoint to our paper in finance is Kyle (1985), who explores trade by a large informed insider. Like us, he explores how liquidity is endogenously fixed in a dynamic environment. His insider trades into a specialist market, whereas ours searches for counterparties. He assumes a diffusion counterparty arrival process, rather than our periodic Poisson arrivals. He computes a dynamic sequential equilibrium, whereas ours simply seeks to optimize optionality. Their insider *depresses the price by selling more* (out of equilibrium), but *our derived supply curve is increasing*. Depth in Kyle (1985) is constant, while our depth monotonically falls in time as the insider unwinds his position.

In summary, we study a pure trading exercise that sidesteps informational issues, indivisibilities, and illiquid-competitive market assumptions. We assume there is no market, and therefore one can only exploit valuable trade opportunities by holding assets. This affords a laser focus on the dynamic tradeoffs in counterparty search.

We present the model in §2, characterize the value function and supply schedule in §3, and perform sensitivity analysis in §4. We analyze liquidity measures in §5, and introduce bargaining in §6. We highlight the recursive logic throughout the paper. Gentle and instructive proofs are in the text, and lengthier ones postponed to the appendix.

2 The Model

Time is continuous on $[0, \infty)$. An infinitely-lived *seller* owns a large *position* $a < \infty$ of a perfectly divisible asset. Each asset share pays him a constant dividend $0 \leq k < \infty$ per unit time, which he discounts at the interest rate $r > 0$.

The asset is illiquid, lacking a formal market. The seller, however, enjoys a flow arrival of buyers, each arriving at a random time with a random offer. Arrivals follow a Poisson process with *arrival rate* $\rho > 0$. In a meeting, the buyer makes a *limit order* (p, x) specifying the share (*bid*) *price* $p > 0$ and *purchase cap* $x > 0$, i.e. the maximum desired quantity. The offer dimensions (P, X) are possibly dependent random variables, with finite mean, from a bounded continuous density $f(p, x)$ that is weakly falling in x . Denote the cdf by $F(p, x)$, expectations w.r.t. f by \mathbb{E} , and marginals by $g(p), h(x) > 0$ on $(0, \infty)$, with $g(p)$ log-concave. The marginal $h(x)$ and conditional expected price $\mathbb{E}[P|x]$ are uniformly bounded, i.e., $h(x) \leq \bar{h} < \infty$, and $\mathbb{E}[P|x] \leq \bar{p} < \infty$ for any $x > 0$.

Our paper simultaneously solves for two possible versions of the model. In the simplest case, the buyer's offer is take-it-or-leave-it. After receiving an offer (p, x) , the seller chooses how much to sell $y \in [0, \min\{x, a\}]$. For instance, he cannot short sell.

Alternatively, the buyer is described by a *reservation price* $w > 0$ and a purchase cap $x > 0$, with density $f(w, x)$. But now the *terms of trade* — price and quantity (p, y) — arise from the Nash bargaining solution. We precisely specify this variation in §6.

After selling quantity $y \leq a$, the seller continues his search with new position $a - y$. He seeks to maximize his expected present value of cash flows from dividends and sales.

3 The Value Function and Selling Strategy

When meeting a buyer, the seller optimally decides whether and how much to act upon the proposed terms of trade. In so doing, he trades off a sure immediate gain for the option value of future trades. We exploit the recursive nature of the problem, and characterize its solution using dynamic programming. Since the position $a \geq 0$ is the natural state variable for the dynamic optimization, the (Bellman) *value* is $V(a)$. Upon meeting a buyer with offer (p, x) , the seller maximizes the present value $py + V(a - y)$ over sales $0 \leq y \leq \min\{x, a\}$. The Bellman equation is $V = \mathcal{T}V$, where:²

$$(\mathcal{T}V)(a) = \frac{ak}{r + \rho} + \frac{\rho}{r + \rho} \mathbb{E} \left(\max_{y \in [0, \min\{X, a\}]} [Py + V(a - y)] \right) \quad (1)$$

²The formula suppresses the randomness of the exponentially-distributed arrival time T of a trader: $(\mathcal{T}V)(a) = \mathbb{E}_T \mathbb{E} \left(\int_0^T ake^{-rs} ds + e^{-rT} \max_{y \in [0, \min\{X, a\}]} [Py + V(a - y)] \right)$.

The seller chooses how much to act upon the proposed terms of trade (p, x) . We call the solution the *supply function* $\mathcal{Y}(p, x, a)$, and prove that it is uniquely defined below.

Re-envision the decision of the seller as a choice of new position $a' = a - y$. Rewrite the maximization in the second capital gain term in the Bellman equation (1) as $pa - \min_{[a - \min\{x, a\}, a]}[pa' - V(a')]$. Consequently, we can think of the seller's problem as minimizing the opportunity cost $pa' - V(a')$ of holding assets. We exploit this logic below.

Note two special cases. If the seller has no option to sell (so that $\rho = 0$), his value reduces to the discounted value of dividends $V(a) = ak/r$. On the other hand, when the asset pays no dividends, the value is a pure option on meeting buyers at arrival rate ρ , whose proposed terms of trade are acceptable. The right hand side of (1) includes a positive probability that the offer is unacceptable, and the seller offers zero supply.

Let \mathcal{B} be the space of bounded continuous functions $V : [0, \infty) \rightarrow \mathbb{R}_+$ with sup norm.

Lemma 1 *\mathcal{T} is a contraction with a unique bounded and continuous fixed point V in \mathcal{B} .*

The proof in the appendix applies Blackwell's sufficient condition for a contraction.

Theorem 1 *The value function $V(a)$ is a strictly increasing and concave function of a .*

Proof: For concavity, let the seller choose the post trade position $z \equiv a - y$. Define the convex constraint set $C(x) = \cup_a \{(z, a) \mid \max\{a - x, 0\} \leq z \leq a\}$. To eliminate the constraint, introduce the characteristic function $\chi_{C(x)}(z, a) = 0$ if $(z, a) \in C(x)$ and $+\infty$ otherwise. So $\chi_{C(x)}$ is convex since $C(x)$ is convex. Rewrite expression (1) for $(\mathcal{T}V)(a)$:

$$(\mathcal{T}V)(a) = \frac{ak}{r + \rho} + \frac{\rho}{r + \rho} \left(a\mathbb{E}[P] - \mathbb{E} \left(\min_{z \geq 0} [Pz - V(z) + \chi_{C(x)}(z, a)] \right) \right) \quad (2)$$

Assume V is concave. Then $pz - V(z) + \chi_{C(x)}(z, a)$ is convex in (z, a) . By Theorem 5.3 of Rockafellar (1970), $\min_{z \geq 0} [pz - V(z) + \chi_{C(x)}(z, a)]$ is convex in a . As expectation preserves concavity, $(\mathcal{T}V)(a)$ is concave in a , and so too is its fixed point $\mathcal{T}V = V$. \square

Observe that the concavity logic is unrelated to standard duality theory in consumer, producer or auction theory — here, *a convex profit function follows from a minimization*.

Standard search models solve a pure stopping exercise, as seen in the left panel of Figure 2, requiring a reservation wage or price. Our seller's supply function is continuously variable. To understand its behavior, we characterize the marginal value function. A concave function is almost everywhere differentiable. We extend this to everywhere. For when V' exists, we show in §A.3 that it solves $V' = \mathcal{S}V'$, where

$$(\mathcal{S}V')(a) = \frac{k}{r + \rho} + \frac{\rho}{r + \rho} \mathbb{E} \left(\max \{ V'(a), \min \{ P, V'(a - \min\{X, a\})(1 + \chi_{[0, a]}(X)) \} \} \right) \quad (3)$$

We prove in §A.3 that \mathcal{S} is a contraction on \mathcal{B} , and so has a unique fixed point V' in \mathcal{B} .

Lemma 2 *The marginal value $V'(a)$ exists on $[0, \infty)$, is continuous, and exceeds k/r .*

We now strengthen Theorem 1 and deduce strict concavity of the value function.

Theorem 2 *The value function $V(a)$ is a strictly concave function of the position a .*

The proof in §A.4 first argues directly that V cannot be linear on any interval $[0, a]$, for the optimal policy would then entail a constant choke price equal to $V'(0+)$; however, that policy would induce a strictly concave value given the purchase caps. The proof then extends this logic to preclude any interval on which the value function is linear.

This result highlights the critical role played by the purchase caps. For absent any caps, the marginal value recursion (3) reduces to the standard Bellman equation for wage search, namely, $rV'(a) = k + \rho \mathbb{E}(\max\{P - V'(a), 0\})$. In this case, the value function is linear $V(a) = aV'(0+)$, and the option to partially unwind the position is worthless.

The position provides valuable trade opportunities, and so the value exceeds their pure dividend value ak/r . While a larger position allows the seller to better avail himself of these options, Theorem 2 asserts that *the optionality value is strictly concave*. In fact, the proof only exploits the finite mean of the offer distribution. For a simple intuition, assume a zero dividend, so that the value entirely owes to the waiting option. A larger position take more time to unwind, and so helps less at the margin.

The marginal value function acts as a marginal cost function for a producer. The seller's supply is likewise inverse to the marginal value function $(V')^{-1}$ (see Figure 1).

Theorem 3 *For any position $a > 0$, share price p , and cap x , the optimal supply is $\mathcal{Y}(p, x, a) = \min\{x, Y(p, a)\}$, where the uncapped supply $Y(p, a)$ is given by:*

$$Y(p, a) = \begin{cases} \max\{a - (V')^{-1}(p), 0\} & \text{for } p \leq V'(0+) \\ a & \text{for } p > V'(0+) \end{cases} \quad (4)$$

Proof: In meetings, the seller solves $\max_y [py + V(a - y)]$ s.t. $0 \leq y \leq x$ and $y \leq a$. As V is strictly concave and the constraints are linear, the FOC is necessary and sufficient for a maximum. Since at most one constraint binds, the constraint qualification for the Kuhn-Tucker conditions is met. If the multipliers are respectively $\lambda_1, \lambda_2, \lambda_3 \geq 0$, then the FOC is $p - V'(a - y) = -\lambda_1 + \lambda_2 + \lambda_3$. By complementary slackness, (i) if $y = x \leq a$, then $p - V'(a - x) \geq 0$, and (ii) if $y = a < x$, then $p - V'(0+) \geq 0$, and (iii) if $y = 0$, then $p - V'(a) \leq 0$. Otherwise, all multipliers vanish, and $p = V'(a - y)$. \square

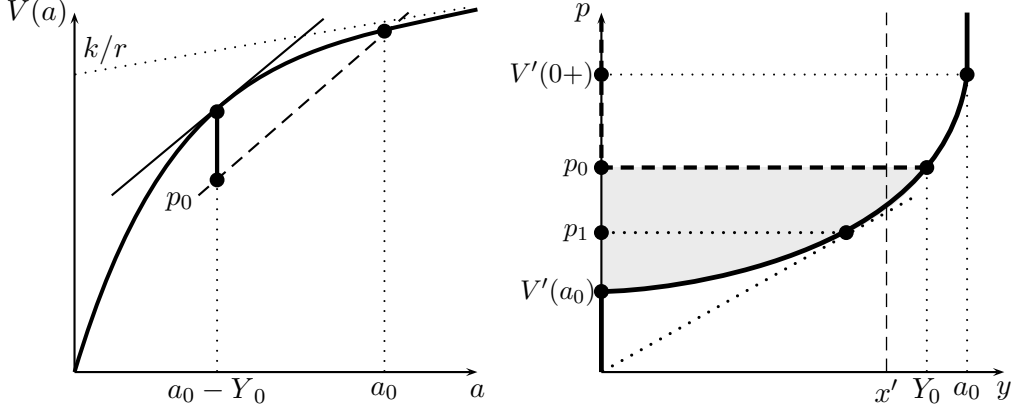


Figure 1: **The value function and inverse uncapped supply.** At left is the increasing and concave value function $V(a)$. Given a limit order price p_0 , surplus is the peak vertical distance of $V(a)$ from the dashed line with slope p_0 . At right, the seller supplies the lesser of the purchase cap x and Y_0 . For the cap x' , the supply is truncated and rises vertically. This optimal supply for the seller maximizes his “producer surplus”, and equates price and the value of the marginal unit $V'(a - y)$. Also, there is a unique secant tangent to Y at some price $p_1 > 0$. By Corollary 1, the supply elasticity exceeds one and is falling for $p < p_1$, and otherwise is less than one.

Notice that never trading is not optimal since this would yield payoff $ak/r < V(a)$, contrary to Lemma 2. Rather the supply (4) is sometimes positive. For any position $a < (V')^{-1}(p)$, the supply function vanishes, but thereafter, it rises dollar for dollar in the position, until it hits the cap x . Equally well, as a function of the purchase cap, supply is either zero (at low prices), or increasing dollar for dollar in the cap until it hits the *uncapped supply* $a - (V')^{-1}(p)$. Next, consider supply as a function of the bid price. Below the *choke price* $V'(a)$, supply vanishes. For higher bid prices that are below the *ask price* $V'(a - x)$, offers are only partially acted upon. For higher bid prices, offers are fully acted upon. Above the *sell-all price* $V'(0+)$, any offer is fully acted upon at *any* position. Curiously, from (3), $V'(0+)$ solves the same Bellman equation as does the value function in a “parallel wage search problem”, namely, $rV'(0+) = k + \rho\mathbb{E}[\max\{P - V'(0+), 0\}]$. So standard search theory mimics ours with divisible assets and a very small position.

With indivisible supply, standard search theory would dictate a choke-price below which no sale occurs. But the optimal sales policy is not constant in time here. Rather, the choke-price reflects the *marginal* value, and the supply is a variable intensive margin that adjusts in the position. From strict concavity, the seller expects a *trade surplus*:

$$\sigma(a) = \mathbb{E}[P\mathcal{Y}(P, X, a) + V(a - \mathcal{Y}(P, X, a)) - V(a)] = \mathbb{E} \max_{y \in [0, \min\{X, a\}]} \int_0^y [P - V'(a - t)] dt \quad (5)$$

So $\sigma(a) > 0$ is the expected gain from a random trade. The Bellman equation for (1) is therefore:

$$rV(a) = ak + \rho\sigma(a) \quad (6)$$

Differentiating (6) yields $rV'(a) - k = \rho\sigma'(a)$, and so $\sigma'(a) > 0$, since $V'(a) > k/r$ by Lemma 2. That $\sigma(a)$ is strictly concave follows from (6) and Theorem 2. Trade surplus $\sigma(a)$ is increasing and strictly concave in the position. Obviously, the value $V(a)$ is proportional to surplus $\sigma(a)$ when $k = 0$. More generally:

Lemma 3 *The search optionality share of value $\sigma(a)/V(a)$ falls in the asset position a , and vanishes as $a \rightarrow \infty$ if $k > 0$. Thus, the limit marginal value is $\lim_{a \rightarrow \infty} V'(a) = k/r$.*

Proof: First, (6) implies $\rho\sigma(a)/V(a) = r - ak/V(a)$. But the secant slope $V(a)/a$ falls in a , as V is increasing and concave, with $V(0) = 0$. So $\sigma(a)/V(a)$ falls in a . Finally, $V'(a) \rightarrow k/r$ iff $V(a)/a \rightarrow k/r$ by L'Hôpital, and by (6), iff $\sigma(a)/V(a) \rightarrow 0$ which is iff $\sigma(a)/a \rightarrow 0$. We prove this limit. The realized trade surplus obeys $py + V(a-y) - V(a) \leq py$. Maximizing over $y \leq \min(x, a)$, and taking expectations in P, X , yields an upper bound $\sigma(a) \leq \mathbb{E}(PX)$. Finally, let $a \rightarrow \infty$ in $0 \leq \sigma(a)/a \leq \mathbb{E}(PX)/a$. \square

So for large asset positions, the seller roughly ignores the search optionality, as his value and trading decisions only reflect the asset dividends. Since the marginal value $V'(a)$ tends to k/r , Theorem 3 implies that trading behavior is asymptotically stationary for $a \rightarrow \infty$, with a choke price k/r . And because $V'(a-x)$ also tends to k/r for fixed x , the optimal supply becomes infinitely elastic for prices near the limit choke price k/r .

To understand how the supply function behaves in the prices, we now characterize the marginal value function, justifying the convex shape in the right panel of Figure 1.

Theorem 4 *The marginal value $V'(a)$ is decreasing and strictly convex in assets a . Moreover, $V''(a) < 0$ exists on $(0, \infty)$, is continuous, and is at least $-(\rho/r)\bar{p}\bar{h}$.*

The seller's value function of the position has the same properties typically assumed for utility functions for money u : increasing, risk averse, and prudent ($u'' < 0 < u', u'''$).

For some insight, assume all trades were accepted. Then the second term would be ρ times the expected price. But the seller passes on low prices $p < V'(a)$, retaining $V'(a)$, while the purchase cap x binds on offers for higher prices $p \geq V'(a-x)$ when $x \leq a$. Since $(r + \rho)V'(a) = k + \rho(V'(a) + \sigma'(a))$ from (6), V' is convex iff $V' + \sigma'$ is. Now:

$$V'(a) + \sigma'(a) = \mathbb{E}[P] + \int_0^\infty \int_0^\infty \max\{V'(a) - p, 0\} dF - \int_0^a \int_0^\infty \max\{p - V'(a-x), 0\} dF \quad (7)$$

The first term on the right side adds the amount that the marginal value exceeds the

price for declined offers. The second term subtracts how much the price exceeds the marginal value for offers with a binding purchase cap. In §A.5, we rewrite (7) using the *uncapped supply* curve $Y(p, a) = a - (V')^{-1}(\min\{p, V'(0+)\})$ derived in (4), we find:

$$V'(a) + \sigma'(a) = \mathbb{E}[P] + \int_0^{V'(a)} F(p, \infty) dp - \int_0^\infty \int_0^{Y(p, a)} \int_p^\infty f(s, t) ds dt dp \quad (8)$$

In the proof of Theorem 4, the first term in (8) is convex, by the recursion assumption. The second term drives the convexity, for its first derivative $-\int_0^\infty \int_p^\infty f(s, Y(p, a)) ds dp$ increases in a . Indeed, the unsigned supply linearly increases in a , and the density of offers $f(p, x)$ is monotone decreasing in the cap x . In other words, for any position, given the derived optimizing behavior, the mass of binding offers — those that cause the diminishing return to optionality — is increasing and concave in the position a . This leads to the convexity of the second two terms for $V'(a) + \sigma'(a)$ in (7), and thus of $V'(a)$.

We now explore the *supply elasticity* $\eta(p, a) \equiv Y_1(p, a)p/Y(p, a)$, as it is expressed in terms of second derivatives of the value function. The convex marginal value is critical.

Corollary 1 *When positive, supply $Y(p, a)$ is increasing and strictly concave in p . The elasticity $\eta(p, a)$ falls in p if $\eta > 1$, and downward single-crosses through 1 (at large a).*

Proof: By Theorem 4, V' is decreasing and convex, and so its inverse is increasing and convex in p . By Theorem 3, the uncapped supply $Y(p, a) = a - (V')^{-1}(p)$ is thus increasing and concave in p , i.e. $Y_1(p, a) > 0$ falls in p . Let $p_1 > 0$ minimize the secant slope $p/Y(p, a)$ on $[0, V'(0+)]$, as in Figure 1. So $Y(p, a)/p$ is increasing in p , and thus $\eta(p, a)$ is falling in p , iff $pY_1 - Y > 0$, and this happens for $p \leq p_1$. For $p > p_1$, the secant is steeper than the tangent in Figure 1, namely, $\eta(p, a) < 1$. We verify that $p_1 < V'(0+)$ for large a , and so there is unit elasticity at a unique point. At the sell-all price (where supply is kinked), the secant slope is $V'(0+)/a$, while the steepest tangent has slope μ , constant in a (as Y has unit slope in a). If $V'(0+)/a \geq \mu$ then $\eta(p, a) \geq 1$ and so $p_1 = V'(0+)$. When $V'(0+)/a < \mu$, then $p_1 < V'(0+)$. Single-crossing follows. \square

For a possibly illuminating retrospective insight into the seller's problem, let us revisit the contrast between our search model and search with indivisible units. With wage search, say, a trader employs a reservation wage, and stops when he secures a higher wage. In our setting, the seller optimally exploits the divisibility, and the average and marginal values diverge, as depicted in Figure 2. His trading strategy is governed by the marginal value. This may be choked off by the buyer (middle panel) or the seller (right panel). Given the concavity, the average always exceeds the margin.

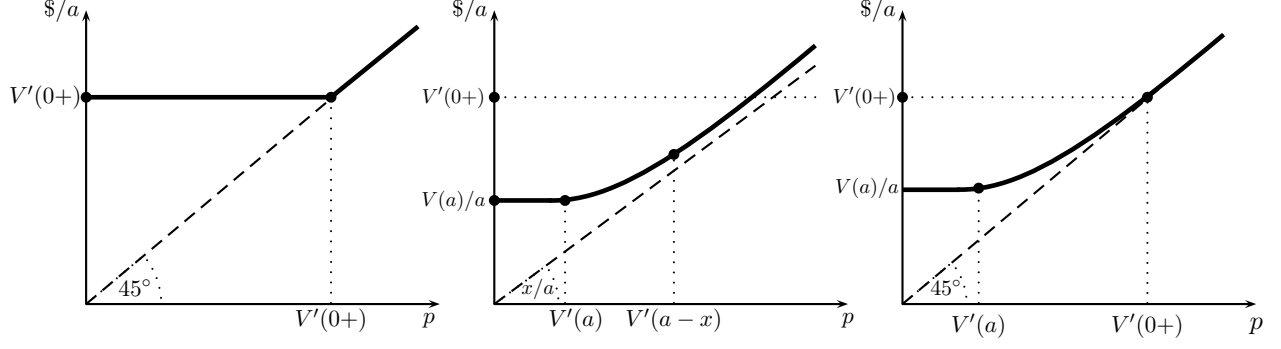


Figure 2: **The Value of Divisibility in Search.** The position is a perpetual call option with zero strike price. The panels plot two functions of the price: the Bellman value per unit of asset (thick lines), i.e., $[p\mathcal{Y}(p, x, a) + V(a - \mathcal{Y}(p, x, a))]/a$, and the intrinsic value of the option (dashed lines): $p \min\{x, a\}/a$. The position is indivisible at left. At the middle and right are the two cases with divisible assets for any price $p < V'(0+)$.

4 Changing Search Frictions and Offer Distributions

With a zero flow dividend, the search friction measure $\psi = r/\rho$ is the only model parameter. More generally, behavior reflects the triple (k, ρ, r) . We now characterize how these model parameters influence the derived functions V, V', V'' . While the value function obeys $V' > 0 > V''$, we now argue that each inequality grows stricter as search frictions diminish: the marginal value naturally rises, but the second derivative falls.

Theorem 5 *For any position $a > 0$, the value $V(a)$ and marginal value $V'(a)$ fall in r and rise in ρ and k , while $V''(a)$ rises in r and falls in ρ and k .*

The comparative statics of V and V' parallel those in the stationary indivisible search model (Figure 3): As search frictions fall, the value increases, and increases faster. But the marginal value falls faster at lower frictions. To wit, the value function flattens.

Since trade surplus $\sigma(a)$ is the maximal area (5) over the marginal value and below the price, it rises in r and falls in ρ and k . Since $rV(a) = ak + \rho\sigma(a)$ by (6), we see that while the value $V(a)$ falls in r , it is less than unit elastic, since $rV(a)$ rises. Equally well, while surplus $\sigma(a)$ falls in ρ , it is less than unit elastic in ρ since $rV(a) = ak + \rho\sigma(a)$ rises in ρ , by Theorem 5. Finally, the value $V(a)$ rises in the dividend k , it is less than unit elastic since $\sigma(a)$ falls in k , given that $V'(a)$ falls, by Theorem 5. Altogether, we can conclude that the share of the value reflecting the search optionality $\rho\sigma(a)/(rV(a)) = 1 - ak/(rV(a))$ falls in the dividend k , but rises in the interest rate r and arrival rate ρ . Counterintuitively, *search optionality matters more with greater impatience, and so asset values proportionately reflect dividends more at lower interest rates*. This paradox arises

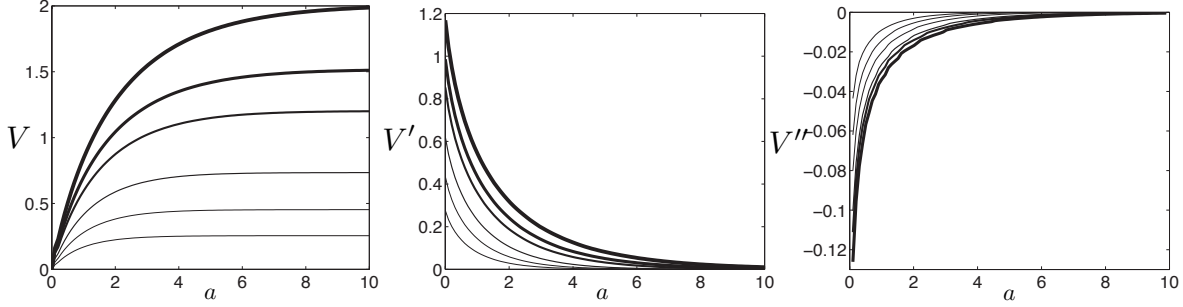


Figure 3: **How Search Frictions Affect $V(a)$, $V'(a)$, $V''(a)$.** We assume no dividends, so that only the value function only depends on $\psi = r/\rho$. From thick to thin lines, search frictions ψ increase: 0.005, 0.1, 0.2, 0.5, 1 and 2. We posit $P \sim \Gamma(1, 1)$ and $X \sim \Gamma(0.5, 1)$, independently.

with indivisible assets since the value V and marginal value V' co-move in Theorem 5, and surplus σ moves inversely to V' .

Next, we explore how shifts in the offer distributions affect the value. We consider changes in the price distribution P conditional on a quantity, fixing the purchase cap marginal $h(x)$, and in the quantity distribution X , conditional on a price, fixing the price marginal $g(p)$. We call these *conditional* stochastic dominance relations.

Theorem 6 *The value V rises with (a) conditional first order stochastic dominance increases in P or X , (b) conditional mean-preserving spreads in P , or (c) conditional second order stochastic order increases in X . The marginal value V' rises with (a).*

As with search theory with indivisible units, like wage search, the seller profits from stochastically better or riskier wages. But in our model, the seller is now *harmed by quantity risk*. Since offers are truncated by his position, he exploits only the lower tail of the purchase cap distribution. In the proof below, we exploit Fenchel duality to show that the purchase caps induce a concave optimized trade payoff for every price.³

Proof of Theorem 6: Write the trade payoff in (1) as $\max_y [py + V(a - y) - \chi_{[0, \min\{x, a\}]}(y)]$, where the argument is concave in y as in the proof of Theorem 1, by concavity of V . The maximum is increasing in p and x . It is also the (Fenchel) conjugate function of $-V(a - y) + \chi_{[0, \min\{x, a\}]}(y)$, and so is convex in p , by Theorem 12.2 of Rockafellar (1970). If the conditionals obey $\tilde{P} \geq_{FSD} P$, or if \tilde{P} is a mean preserving spread of P for all x :

$$\mathbb{E}_{\tilde{P}} \left(\max_{0 \leq y \leq \min\{x, a\}} (\tilde{P}y + V(a - y)) | x \right) \geq \mathbb{E}_P \left(\max_{0 \leq y \leq \min\{x, a\}} (Py + V(a - y)) | x \right) \quad (9)$$

³It is natural to ponder the effect of risk on the marginal value, and thereby on trading behavior. All our numerical simulations suggest that V' falls with mean preserving spreads in the purchase caps X . We have been unable to formally establish this recursively, since the right side of (3) is not concave.

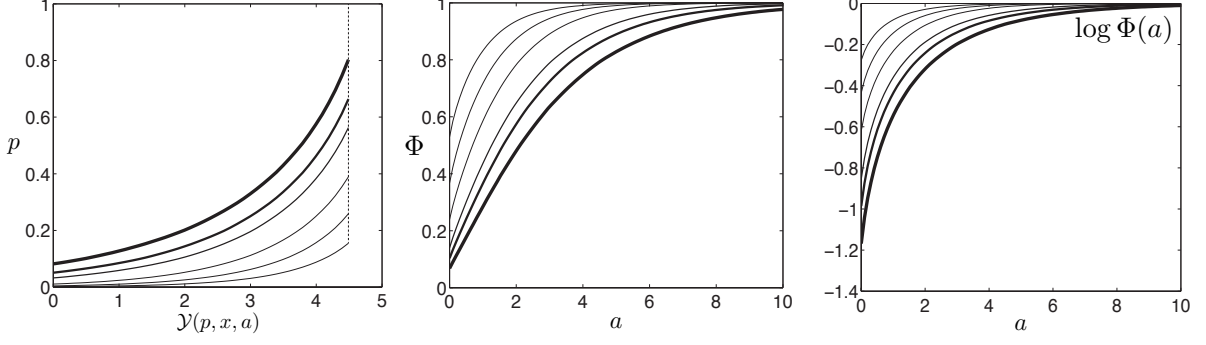


Figure 4: **How Search Frictions Affect Supply and the Trade Chance.** For $P \sim \Gamma(1, 1)$ and $X \sim \Gamma(0.5, 1)$ and $k = 0$. Left: The supply for $a = 5$ and $x = 4.4$. Center: The trade chance as a function of the position. Right: The log of the trade chance as a function of assets. In all cases frictions $\psi = r/\rho$ increase from thick to thin lines: 0.005, 0.1, 0.2, 0.5, 1 and 2.

Define the Bellman operator $\tilde{\mathcal{T}}$ for (\tilde{P}, X) . Since $\tilde{h}(x) = h(x)$, the expectation of (9) in P yields $(\tilde{\mathcal{T}}V)(a) \geq (\mathcal{T}V)(a)$, and so the fixed points obey $\tilde{V} \geq V$.

Similarly, the trade payoff is also concave in x . For since the constraint set $C(a) = \{(y, x) | 0 \leq y \leq \min\{x, a\}\}$ is convex in (y, x) , the characteristic function $\chi_{C(a)}(y, x)$ is convex in (y, x) . So $-\min_{y \geq 0}[-py - V(a - y) + \chi_{C(a)}(y, x)]$ is concave in x , as in the proof of Theorem 1. So the fixed point rises in FSD and SSD shifts in X .

Lastly, $\max\{V'(a), \min\{p, V'(a - \min\{x, a\})(1 + \chi_{[0, a]}(x))\}\}$ in (3) rises in p and x . So its expectation V' rises with first order stochastic dominance increases in P, X . \square

5 Trading Behavior and Liquidity Measures

If a buyer offers more generous terms of trade, the seller is willing to part with more of his position (Figure 1). This *transactional liquidity* reflects the seller's forward-looking and nonstationary behavior as he trades off sure money today and possible money tomorrow. But given the exponential arrivals, our search model affords a second liquidity measure: While the arrival rates of opportunities is fixed, the arrival rate of desirable offers is endogenous and equals $\rho\Phi(a) = \rho(1 - F(V'(a), \infty))$, where $\Phi(a)$ is the *trade chance*. The expected time to trade is therefore $\tau(a) = 1/(\rho\Phi(a))$ and its variance $\xi(a) = 1/(\rho\Phi(a))^2$. Since both fall in the trade chance, with a smaller position, higher dividends or a lower interest rate, the seller is less eager to sell, and the mean and variance of trade times accordingly increase. But with a higher arrival rate ρ , a tradeoff emerges: The seller can afford to hold out for better offers, but they come more often. Given the log-concave price distribution, the first effect does not overwhelm the second, at least for small ρ .

Theorem 7 *The expected time to trade $\tau(a)$ and its variance $\xi(a)$ are decreasing and log-concave in a . They fall in r , rise in k , and fall in ρ , when $\rho \leq \bar{\rho}$.*

Proof: So it suffices that $\Phi(a)$ increase in r , decrease in k , and be increasing and log-concave in a . Now, $V'(a)$ falls in a by Theorem 2, and falls in r and increases in k by Theorem 5. So the trade chance $\Phi(a) = 1 - F(V'(a), \infty)$ increases in a and r , but falls in k . Since the epigraph $\{(p, a) | p \geq V'(a)\}$ of the convex function V' is convex, its indicator function is log-concave. Because the marginal $g(p)$ is assumed log-concave, the trade chance written as $\Phi(a) = \int_0^\infty g(p) \mathbb{1}_{\{p \geq V'(a)\}} dp$ is log-concave, by Prékopa (1973). \square

A lesson of Theorem 7 is that the seller finds it increasingly hard to trade as he unwinds his position, and it grows harder to predict the next trade time. But this is even stronger than stated. For given the log-convexity result in Theorem 7, not only do the mean waiting time $\tau(a)$ and variance $\xi(a)$ increase as the asset is sold off, but they *proportionately* increase — for instance, $-\tau'(a)/\tau(a)$ rises as a falls.

The proof of Theorem 7 in §A.7 establishes more than is claimed. In particular, it shows that the elasticity $|\mathcal{E}_\rho(\Phi(a))|$ falls in the position: The trade chance falls in the meeting rate ρ proportionally more at smaller positions. This monotone elasticity claim also holds for the interest rate and dividend. Namely, both $|\mathcal{E}_r(\Phi(a))| = \varphi(V'(a))(-V'_r(a))r$ and $|\mathcal{E}_k(\Phi(a))| = \varphi(V'(a))V'_k(a)k$ are decreasing in the position. That trading behavior is less responsive to search frictions and dividends at larger positions is consistent with the diminishing search optionality found in Lemma 3. In fact, all elasticities are less than unity at high search frictions — namely, large r or small ρ .

Let's now consider transactional liquidity. First, let's consider predictions about trade prices. Since the expected trade price $\mathbb{E}(P | P \geq V'(a))$ falls in the position a , by Theorem 1, the seller holds out for better prices while unwinding his position. But as seen in Figure 5, given the log-concave price density $g(p)$, the variance of traded prices $\text{Var}(P | P \geq V'(a))$ is nondecreasing in the position a (Heckman and Honoré, 1990). So as the seller sells his position, his terms of trade improve and grow more predictable, and the *expected mark-up* $\mathbb{E}[P - V'(a) | P \geq V'(a)]$ falls.⁴

Finally, we consider transactional liquidity from finance relating price and quantity. Our seller is analogous to the long-lived insider in Kyle (1985), whom we will assume, for definiteness, wishes to sell. His derives a trading rule that optimally trades off exploiting his informational edge and securing its fruits. In solving his dynamic arbitrage, he finds that the *market depth* — namely, the absolute slope of the demand curve — is optimally

⁴By a definition of Keynes (1930), the asset is more liquid with a smaller position: “One asset is more liquid than another one if it is more certainly convertible into money at short notice without loss.”

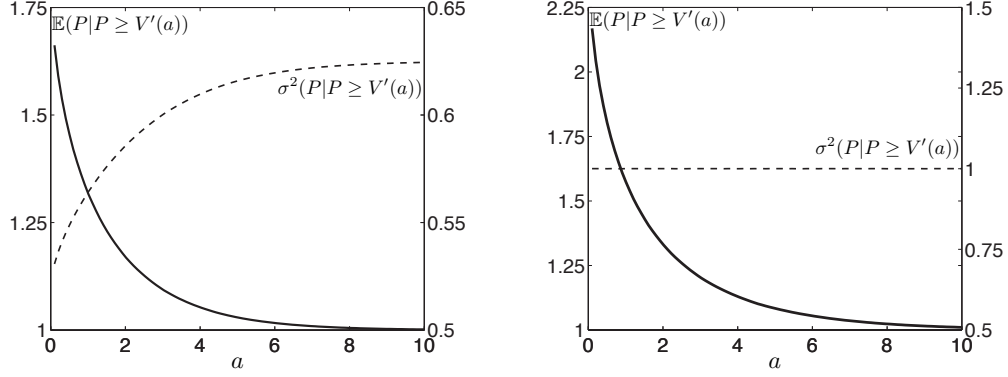


Figure 5: **Expected price and variance while unwinding.** The expected trade price increases and variance falls. At left, when $P \sim \Gamma(1.6, 1/1.6)$, so $\mathbb{E}(P) = 1$ and $s(P) = 0.625$. At right, when $P \sim \Gamma(1, 1)$ $\mathbb{E}(P) = 1$ and $s(P) = 1$, that remains unchanged. Both cases assume $X \sim \Gamma(0.5, 1)$.

constant in time and price. Our seller also has a dynamically optimal selling schedule that trades off price and quantity at each instant, and we likewise can use this depth measure. But here we see the critical role of Kyle's informational asymmetry: *For his insider depresses the price by selling more today, whereas our seller throws away future options by selling more.* Our seller only sells more when offered a higher price, a standard assumption of supply curves.⁵ In our setting, depth is the reciprocal slope of the inverse supply curve $\lambda(y, a) = -1/V''(a - y)$. Unlike Kyle's competitive model, depth is not constant: It increases in the position and falls in the trade size, by Theorem 4.

We can also address a property of liquid markets. For trades $y < a$, our seller charges a *purchase premium* $\pi(y, a) = V'(a - y) - V'(a)$ over his choke price, since his search optionality is proportionately more valuable with a smaller position. Analogous to a tightness measure, this premium is intuitively smaller in a more liquid market, since a trader has a smaller price impact.⁶ We next explore measures of transactional liquidity.

Theorem 8 (a) Supply $\mathcal{V}(p, x, a)$ is nondecreasing in a, r , and nonincreasing in p, k .
(b) The supply elasticity $\eta(p, a)$ is decreasing and convex in a , and vanishes as $a \rightarrow \infty$. Depth $\lambda(y, a)$ is increasing in a and decreasing in y . It falls in p and k , and rises in r .
(c) The purchase premium $\pi(y, a)$ is increasing in the trade size $y < a$ and decreasing in assets a . It falls in r , and rises in p and k .

As the position a explodes, the optimal sales policy converges to a stationary rule — the seller avails himself fully of all limit offers with prices $p \geq k/r$, and otherwise abstains. Indeed, $V'(a) \rightarrow k/r$ as $a \rightarrow \infty$, by Lemma 3. But sales stochastically drift

⁵A different dynamic tradeoff arises in each case: Kyle's seller optimally exploits his informational edge over time (Anderson and Smith, 2013), whereas ours optimally exploits his search optionality.

⁶In a two sided version of this paper that we have begun, this will analogize to a bid-ask spread.

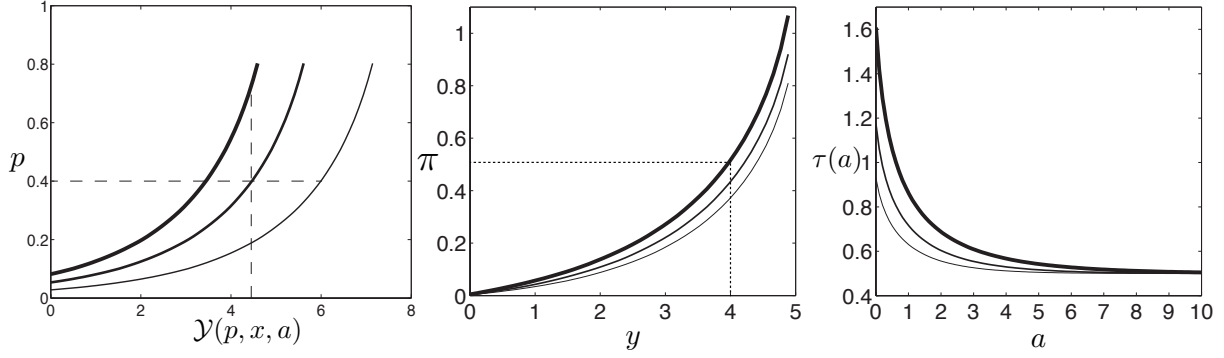


Figure 6: **Transactional Liquidity and Time to Trade.** When $k = 0$ and $a = 5$. At left is the elasticity $\eta(p, a)$ through the horizontal dashed line, and depth $\lambda(y, a)$ through the vertical one. At center, the purchase premium as a function of trade size. The right panel depicts the expected time to trade $\tau(a)$ as a function of a . In the center and right panels, frictions $\psi = r/\rho$ increase from thick to thin lines: 0.005, 0.1 and 0.5.

down as the seller's position unwinds. For the seller's own cap starts to bind more than the purchase caps, and he simultaneously grows more choosy due to value concavity — e.g. his choke price rises. A nearly stationary rule is once again optimal for small positions a , selling out for any price $p > V'(0+)$, and the purchase caps don't bind.

Likewise, as the asset position falls, the seller's optionality value figures more prominently in his optimization (Lemma 3). Accordingly, his purchase premium rises and his depth falls — the ask-price grows more responsive to the trades. This suggests that as time passes, liquidity worsens. Yet paradoxically, this supply elasticity rises (Figure 6). As the position falls to zero, the seller exploits the divisibility of the asset less, and his prices all converges to the sell-all price.

It might seem intuitive that liquidity worsens with greater search frictions. We find, quite the contrary, that *transactional liquidity improves with search frictions* — i.e., a higher interest rate or a lower arrival rate.⁷ Theorem 8 asserts that the seller's chosen trade volume⁸ and depth rises, and the purchase premium falls. On the other hand, more search frictions worsens one transactional liquidity measure — trade prices fall and grow more volatile, and the expected markup rises.⁹ For V' falls, by Theorem 5.

The new dimension of liquidity afforded by this search framework, the waiting time between transactions, has an ambivalent response to increased search frictions. With

⁷E.g., trading volume rises at higher interest rates (Proposition 2 in Anderson and Smith (2013)).

⁸ Trade volume does not fall too fast, for we show in §A.8 that the expected sales rate $\rho \mathbb{E}[\mathcal{Y}(P, X, a)]$ increases in the arrival rate ρ , when ρ is low.

⁹It might seem puzzling that the expected markup $\mathbb{E}[P - V'(a) | P \geq V'(a)]$ rises and yet the purchase premium falls. But the seller's first order condition $p - V'(a - y) = 0$ only holds for interior solutions. Absent purchase caps, it would be an identity, and the expected markup equals the premium $\pi(y, a)$.

lower interest rates, by Theorem 7, sales slow down, and the seller waits longer to trade.¹⁰ But with a higher arrival rate ρ , sales accelerate, and so waiting times fall.

6 Search and Nash Bargaining

The search framework naturally lends itself to answering questions about bargaining power that might influence liquidity. For trade opportunities only arrive periodically, and are intuitively subject to negotiation. One might wonder whether our theory still applies. We argue that it extends when the Nash bargaining solution describes negotiation.

Assume bargaining weights $\delta \in [0, 1]$ and $1 - \delta$ on the surplus of the seller and buyer, respectively. The seller's surplus of trading (over not trading) is $s(p, y, a) = py + \mathcal{V}(a - y) - \mathcal{V}(a)$, and the buyer's surplus is $(w - p)y$. The terms of trade dictated by the Nash solution entail a *negotiated price* \mathcal{P} and *bargained supply* Υ functions obeying:

$$(\mathcal{P}(w, x, a), \Upsilon(w, x, a)) \equiv \arg \max_{\{p, 0 \leq y \leq \min\{x, a\}\}} s(p, y, a)^\delta ((w - p)y)^{1-\delta} \quad (10)$$

We solve this maximization in stages. The FOC in p suffices by concavity in p . Given the reservation offer (w, x) , the negotiated price is the weighted average of the two parties' reservation prices:

$$p = \delta w + (1 - \delta)(\mathcal{V}(a) - \mathcal{V}(a - y))/y \quad (11)$$

The seller and buyer respectively secure fractions δ and $1 - \delta$ of the *total surplus* $\mathcal{S}(w, x, a)$. The bargained supply must maximize total surplus, namely, $\Upsilon(w, x, a) = \arg \max_{y \in [0, \min\{x, a\}]} \mathcal{S}(w, x, a)$, exactly as in our original model. We offer some insights:

1. *The Nash bargaining model is formally equivalent to the original model with a lower arrival rate $\rho\delta$ of offers (w, x) drawn from the density f .* For since the seller is risk neutral, we can just as well imagine that he secures price w with chance δ and otherwise gets his reservation (zero surplus) price. We recover our original model with $\delta = 1$. The case $\delta = 0$ erases all trade surplus, and the seller holds assets for their dividend stream, i.e. $\mathcal{V}(a) = ak/r$. In other words, greater buyer bargaining power means more frictions.

2. *The seller's value, marginal value and absolute second derivative are lower, since $\mathcal{V}(a|\rho, \delta) \equiv V(a|\delta\rho)$ and $\mathcal{V}'(a|\rho, \delta) \equiv V'(a|\delta\rho)$, and recalling Theorem 5.*

3. *Greater bargaining power for buyers raises supply and lowers the negotiated price, the choke price, and the sell-all price:* The bargained supply $\Upsilon(w, x, a)$ is given by (4)

¹⁰Our liquidation model sheds light on Alan Greenspan's comment: "Super slow interest rates can actually slow the process of liquidation, because the cost of carrying debt is so low" (Leonard and Coy, 2012). As argued in §3, the seller minimizes the opportunity cost $pa' - V(a')$ of holding a position a' .

but with meeting rate $\rho\delta$. By Theorem 8(a), it falls in the seller's bargaining power δ . Next, as the buyer secures a fraction $1 - \delta$ of total surplus $\mathcal{S}(w, x, a)$, we have

$$[w - \mathcal{P}(w, x, a)]\Upsilon(w, x, a) = (1 - \delta)\mathcal{S}(w, x, a) \quad (12)$$

Recall that $\mathcal{S}(w, x, a) = \max_{y \in [0, \min\{x, a\}]} \int_0^y (p - \mathcal{V}'(a - t))dt$ falls in δ by Theorem 5. In the corner solution when $\Upsilon(w, x, a) = \min\{x, a\}$, the price $\mathcal{P}(w, x, a)$ rises in δ . We now claim that this holds generally when $\Upsilon(w, x, a) < \min\{x, a\}$ and $w \equiv \mathcal{V}'(a - \Upsilon(w, x, a))$. For define the trade surplus $s(y, a) \equiv \mathcal{V}(a - y) + \mathcal{V}'(a - y)y - \mathcal{V}(a)$, and rewrite (12) as $\mathcal{P}(w, x, a) = w - (1 - \delta)s(\Upsilon(w, x, a), a)/\Upsilon(w, x, a)$. Appendix Lemma A.1 verifies that $s(y, a)/y$ rises in y . Thus, $s(\Upsilon(w, x, a), a)/\Upsilon(w, x, a)$ falls in δ , since $\Upsilon(w, x, a)$ falls in δ . Finally, the two threshold prices fall as $\mathcal{V}'(a) < \mathcal{V}'(a)$.

The logic of this point implies that with greater search frictions, not only does the bargained supply increase (as is true without bargaining), but the negotiated price falls.

4. *Not only does bargained supply rise in the position, but the negotiated price falls.* Supply rises just as in (4). Next, substitute the optimal supply $y = \Upsilon(w, x, a)$ into (10). This reduces to (*): $\Upsilon(w, x, a)(p - c(\Upsilon(w, x, a), a))^\delta(w - p)^{1-\delta}$, where the secant slope of \mathcal{V} is

$$c(y, a) = [\mathcal{V}(a) - \mathcal{V}(a - y)]/y = \int_0^1 \mathcal{V}'(a - (1 - t)y)dt \quad (13)$$

First, by Topkis (1978), the price $\mathcal{P}(w, x, a)$ rises in a since (*) is log-supermodular in $(p, -a)$ — as its middle factor $(p - c(\Upsilon(w, x, a), a))^\delta$ is log-supermodular in $(p, -a)$. For the slope of supply $\Upsilon(w, x, a)$ in a is at most one, as in (4). So substituting $y = \Upsilon(w, x, a)$ in (13), the argument $a - (1 - t)y$ rises in a , i.e. $c(\Upsilon(w, x, a), a)$ falls in a , as \mathcal{V} is concave.

5. *The trade value is increasing and concave in the position a until the purchase cap binds, and then decreasing and convex.* Without bargaining, the trade value plot perfectly resembles the supply (4), since the price is fixed — it is piecewise linear in the position a , rising with slope w until $\mathcal{V}'(a - \min\{x, a\}) = w$, and then is constant. With bargaining, the trade value $\mathcal{P}(w, x, a)\Upsilon(w, x, a)$ initially vanishes, then is increasing and *strictly* concave in a until $\mathcal{V}'(a - \min\{x, a\}) = w$, and thereafter decreasing and strictly convex.¹¹ For supply is fixed at x , but the price is decreasing and strictly convex in a .

6. *Bargaining lowers the trade value except for a low reservation price and position.* For high reservation values w above the sell-all price $\mathcal{V}'(0+) > \mathcal{V}'(0+)$, supply is unchanged and the price is lower, so the trade value is lower. Next, consider lower w . The

¹¹Indeed, by (11), the trade value is $\delta w \Upsilon(w, x, a) + (1 - \delta)(\mathcal{V}(a) - \mathcal{V}(a - \Upsilon(w, x, a)))$, and $a - \Upsilon(w, x, a)$ is constant in a when the purchase cap does not bind, by (4). Finally, $\Upsilon(w, x, a)$ is piecewise linear, and $\mathcal{V}'(a)$ falls and is strictly convex.

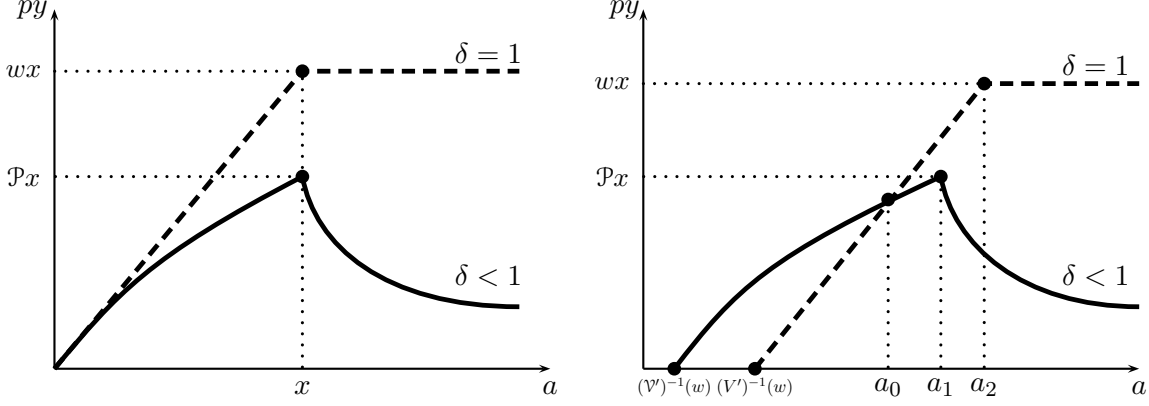


Figure 7: **The effect of bargaining on the trade value.** In both panels, the thick and dashed lines plot the trade value with and without bargaining respectively. At left, when $w > V'(0+)$, $w\mathcal{Y}(w, x, a) > \mathcal{P}(w, x, a)\Upsilon(w, x, a)$. At right, when $w \leq V'(0+)$, the trade value raises for low positions $a \leq a_0$. At the asset positions a_1 and a_2 the purchase cap binds with bargaining and without bargaining respectively.

bargained supply $\Upsilon(w, x, a)$ has unit slope in a , and surplus $\mathcal{S}(w, x, a)$ rises in a . Thus, from (12), the trade value $\mathcal{P}(w, x, a)\Upsilon(w, x, a)$ has slope at most w in a . But the slope in a of the trade value $w\mathcal{Y}(w, x, a)$ without bargaining equals w , recalling Theorem 3. Since the maximum of $\mathcal{P}(w, x, a)\Upsilon(w, x, a)$ lies below its no-bargaining counterpart wx , and it falls after peaking, the trade values must cross, as depicted in Figure 7.

7. *Greater bargaining power for buyers lowers the mean and variance of waiting times.* As in Theorem 7, this follows because the chance of a desirable trade $(1 - F(\mathcal{V}'(a), \infty))$ is now higher — because \mathcal{V}' rises in ρ and thus in δ , by Theorem 5.

8. *Greater bargaining power for buyers increases depth and lowers the purchase premium.* For given the FOC $w \equiv \mathcal{V}'(a - y)$, the *inverse uncapped supply curve* (11) is:

$$p(y, a) = \delta \mathcal{V}'(a - y) + (1 - \delta) \int_0^y \mathcal{V}'(a - t) dt / y \quad (14)$$

Firstly, depth is the inverse slope $\Lambda(y, a) = (\partial p(y, a) / \partial y)^{-1}$, and the uncapped supply has slope $p_1(y, a) = -\delta \mathcal{V}''(a - y) + (1 - \delta) c_1(y, a)$, where $c_1(y, a) = -y^{-2} \int_0^y \int_{a-y}^{a-t} \mathcal{V}''(u) du dt$, recalling (13). Next, using (14), rewrite the purchase premium $\Pi(y, a) = p(y, a) - \mathcal{V}'(a)$ as

$$\Pi(y, a)y = \delta \int_0^y (\mathcal{V}'(a - y) - \mathcal{V}'(a - t)) dt + \int_0^y (\mathcal{V}'(a - t) - \mathcal{V}'(a)) dt$$

By our equivalence result, it suffices that \mathcal{V}'' fall in ρ and thus in δ (true by Theorem 5).

9. *The qualitative behavior of market depth, the purchase premium, and the elasticity claimed in Theorem 8 still hold with bargaining.* This is verified in Appendix A.9.

7 Conclusions

In this paper, we extend search theory to the sale of perfectly divisible units. We assume that an owner of a large asset position faces the periodic arrival of counterparties demanding variable quantities of an asset. Inasmuch as the counterparty arrival process in finance is better approximated by periodic arrivals than a constant flow, then this search theory story of the liquidation process is a useful alternative framework for understanding liquidity. This model explains, e.g., why and how a seller might sell more to a buyer seeking 100 units than to a contemporaneous buyer desiring 1000 units.

Our findings owe to a novel structure of the Bellman value function: Using methods in convex duality theory and recursion, we derive an increasing and concave value function — diminishing returns to optionality — with a convex marginal value function. As the seller’s asset position vanishes, search optionality matters more, and thus he grows choosier. In the limit, the choke price converges to the sell-all price, and his selling policy converges to that with indivisible assets. The model offers new insights on transactional liquidity, and suggests a new dimension of liquidity — the waiting time between trades. The model is readily amenable to Nash bargaining.

One might think that demand curves facing the seller might perchance be more richly described than the limit orders assumed here. But these have an established tradition in finance, and we know of no tractable random space of such demand curves.

We have also analyzed a “pure” problem with a homogeneous asset, having a scalar summary. With many asset classes, one should apply our theory separately for each class.

Since our model is a decision theory exercise, we hope that it can be a key ingredient in equilibrium analyses in which the buyer’s behavior is derived and not exogenously specified. We are now extending the analysis here to a middleman facing an inventory management problem, with periodic arrivals of *buyers or sellers* with trade offers.

A Appendix: Omitted Proofs

A.1 The Bellman Operator: Proof of Lemma 1

Take $V(a) \in \mathcal{B}$ with $V(a) < ak/r + \rho\mathbb{E}[PX]/r$ and use (1) to show $(\mathcal{T}V)(a) \leq ak/r + \rho\mathbb{E}[PX]/r$. The maximum of a continuous function $py + V(a - y)$ yields a continuous function in a . Since $f(p, x) \in \mathcal{B}$, then $\mathcal{T}:\mathcal{B} \rightarrow \mathcal{B}$. We check that \mathcal{T} meets the Blackwell sufficient conditions for a contraction. For monotonicity of \mathcal{T} , take $W \in \mathcal{B}$ with $V(a) \leq W(a)$. Easily, $(\mathcal{T}V)(a) \leq (\mathcal{T}W)(a)$. Likewise, $(\mathcal{T}V + b)(a) < (\mathcal{T}V)(a) + \beta b$, where

$\beta = \rho/(r + \rho) < 1$. Since \mathcal{B} with the sup norm is a complete metric space, by the Contraction Mapping Theorem, it has a unique fixed point $\mathcal{T}V = V \in \mathcal{B}$. \square

A.2 Monotonicity and Concavity: Proof of Theorem 1

Since \mathcal{T} is a contraction, and monotonicity are closed properties in \mathcal{B} with the sup norm, it suffices to show that is preserved by \mathcal{T} (Corollary 3.2.1 in Lucas, Stokey, and Prescott (1989)). From (1), since the choice set and the objective function increase in a , then $(\mathcal{T}V)(a) \leq (\mathcal{T}V)(a')$, so the fixed point increases in a . Second, $V(a)$ is strictly increasing, for one sales strategy available at $b > a$ is to act as if one's position is a , and for unexploited offers $\bar{F}(0, a) > 0$ sell at any positive price, which yields extra payoff at least $\mathbb{E}[P \min\{b - a, X - \min\{X, a\}\}] > 0$. \square

A.3 Differentiability of the Value Function: Proof of Lemma 2

We show that there exists a continuous function V' satisfying (3), and that any solution of (3) is the derivative of (1). We attack these tasks in reverse order.

STEP 1. If the value function is continuously differentiable on $[0, \infty)$, then Theorem 3 is valid. (Its proof only exploits weak concavity of V .) So applying Corollary 5 in Milgrom and Segal (2002) to (1) is valid, and yields for the second term $\mathbb{E}(V'(a - \mathcal{Y}(p, x, a)) + \lambda_2(\mathcal{Y}(p, x, a)))$. This is the derivative $V'(a)$ for $p \leq V'(a)$, p for $V'(a) \leq p \leq V'(a - \min\{x, a\})$ and $\min\{p, V'(a - \min\{x, a\})(1 + \chi_{[0, a]}(x))\}$ otherwise. So (3) follows from adding $k/(r + \rho)$ to this derivative.

STEP 2. By standard logic, we will show that the \mathcal{S} operator in (3) is a contraction, and so has a unique bounded and continuous fixed point V' . We can use (3) to show $0 \leq (\mathcal{S}V')(a) \leq k/r + \rho\mathbb{E}[P]/r$ whenever $0 \leq V'(a) \leq k/r + \rho\mathbb{E}[P]/r$, so that $\mathcal{S}: \mathcal{B} \rightarrow \mathcal{B}$. Next, by the Maximum Theorem, the right side of (3) is continuous in a . Since $f(p, x) \in \mathcal{B}$, we have $\mathcal{S}: \mathcal{B} \rightarrow \mathcal{B}$. We check that \mathcal{S} obeys the Blackwell sufficient conditions for a contraction. To see monotonicity of \mathcal{S} , take $W \in \mathcal{B}$ with $V'(a) \leq W(a)$. Since the right side of (3) is monotone, $(\mathcal{S}V')(a) \leq (\mathcal{S}W)(a)$. Easily, $(\mathcal{S}V' + b)(a) < (\mathcal{S}V')(a) + \beta b$, where $\beta = \rho/(r + \rho) < 1$. So \mathcal{S} has a unique fixed point $\mathcal{S}(V') = V' \in \mathcal{B}$. Finally, to see $V'(a) > k/r$, by concavity and $V'(0+) < \infty$, we have $0 < \bar{F}(V'(0+), 0) \leq \bar{F}(V'(a), 0)$. Hence, $V'(a) = (\mathcal{S}V')(a) > k/(r + \rho) + \rho V'(a)/(r + \rho)$, so $V'(a) \geq k/r$. \square

A.4 Strict Concavity: Proof of Theorem 2

By Theorem 1, it suffices that V is never linear on an interval. Assume this is false, and now choose the least interval $I=[a_1, a_2]$ on which V is linear, say with slope $c \in (0, \infty)$.

Assume first $a_1 = 0$, so that $V(a) = ca$ on I . The optimal policy in (1) for position $a \in I$ is thus to sell $y = \min\{x, a\}$ if $p \geq c$, and $y = 0$ otherwise. Define the *survivor* $\bar{F}(p, x) = \int_x^\infty \int_p^\infty f(s, t) ds dt$ and the cdf $F(p, x) = \int_0^x \int_0^p f(s, t) ds dt$. Then (1) asserts

$$c(r + \rho)a \equiv a \left(k + \rho c + \rho \int_c^\infty \bar{F}(p, 0) dp \right) - \rho \int_0^a \int_c^\infty [1 - \bar{F}(0, x) - F(p, x)] dp dx \quad (15)$$

identically in interval I . Consequently, the second derivative of right side of (15) vanishes — namely, $\rho \int_c^\infty \int_p^\infty f(s, a) ds dp < 0$. This implies $c = \infty$, a contradiction.

Next assume $a_1 > 0$. Now exploit (3) to get for any position $a \in I$,

$$c = \frac{k}{r + \rho} + \frac{\rho}{r + \rho} \int_0^\infty \int_0^\infty \max \left\{ c, \min \left\{ p, V'(a - \min\{x, a\})(1 + \chi_{[0, a]}(x)) \right\} \right\} dF(p, x) \quad (16)$$

Let $U(a)$ be the right side of (16). Write the max term in $U(a_i)$ as $\max(c, \min(p, \gamma_i \zeta_i))$ for $i = 1, 2$. Since $a_2 > a_1$, we have $\gamma_2 \leq \gamma_1$ by concavity of V , and $\zeta_2 \leq \zeta_1$. Thus,

$$\max(c, \min(p, \gamma_2 \zeta_2)) - \max(c, \min(p, \gamma_1 \zeta_1)) \leq \max(c, \min(p, \gamma_1 \zeta_2)) - \max(c, \min(p, \gamma_1 \zeta_1))$$

The only non-zero terms on the right side arise for $\zeta_1 > \zeta_2$ — i.e. $a_1 < x < a_2$ — and $p > V'(0+)$. Taking integrals, $V'(a_2) - V'(a_1) = \rho \int_{a_1}^{a_2} \int_{V'(0+)}^\infty [V'(0+) - p] dF(p, x) < 0$. \square

A.5 Convex Marginal Value: Proof of Theorem 4

Write the right side of the V' operator in (3) as $(\mathcal{S}V')(a) = k + \beta(V'(a) + \sigma'(a))$ where $\beta = \rho/(r + \rho)$. We now reformulate the expression for $V'(a) + \sigma'(a)$ in (7). Integrate by parts using $\int_0^u (u - x) dF(x) = \int_0^u F(x) dx$ and $\int_b^\infty (x - b) d\bar{F}(x) = - \int_b^\infty \bar{F}(x) dx$, and change the order of integration. Use the unsigned supply curve $Y(p, a) = a - (V')^{-1}(\min\{p, V'(0+)\})$ that removes the choke price from (7). This implies:

$$(\mathcal{S}V')(a) = \frac{k}{r + \rho} + \beta \left(\mathbb{E}[P] + \int_0^{V'(a)} F(p, \infty) dp - \int_0^\infty \int_0^{Y(p, a)} \int_p^\infty f(s, t) ds dt dp \right) \quad (17)$$

Now assume that $V'(a)$ is convex in a . Since $\int_0^u F(p, \infty) dp$ is increasing and convex in u , by Theorem 5.1 in Rockafellar (1970), $\int_0^{V'(a)} F(p, \infty) dp$ is convex in a . Since

$Y_2(p, a) \equiv 1$, the second integral in (17) is also convex in a since its derivative in a , namely $-\int_0^\infty \int_p^\infty f(s, Y(p, a)) ds dp$, is increasing in a . In summary, \mathcal{S} preserves convexity, a closed property under the sup norm. So the fixed point $\mathcal{S}(V') = V'$ is convex in a .

To prove strict convexity, from the second conclusion of Corollary 3.2.1 in Lucas, Stokey, and Prescott (1989), it suffices that the image of a convex function V' is strictly convex. This follows because $\int_0^\infty \int_p^\infty f(s, Y(p, a)) ds dp$ strictly falls in a since $f(s, y)$ must eventually strictly fall in y , as $y \rightarrow \infty$, or the expected price would be infinite.

We now show that V' is differentiable in a , paralleling the proof of Lemma 2 in §A.3.

STEP 1. Assume V' is continuously differentiable on $[0, \infty)$. Differentiating (3):

$$V''(a) = \beta \left(\int_{V'(0+)}^\infty (V'(0+) - p) f(p, a) dp + F(V'(a), \infty) V''(a) + \int_0^a \int_{V'(a-x)}^\infty V''(a-x) dF \right) \quad (18)$$

Any solution V'' of (18) is the derivative of V' , if it is differentiable. As in §A.3, it suffices that for any given V' , that exists a continuous function V'' satisfying (18).

STEP 2. We show that the right side of (18) defines a contraction mapping \mathcal{H} , and so has a unique bounded and continuous fixed point V'' , with $-(\rho/r)\bar{p}\bar{h} \leq V''(a) \leq 0$ for all $a > 0$. Take $V'' \in \mathcal{B}$ with $-(\rho/r)\bar{p}\bar{h} \leq V''(a) \leq 0$. Since $V'(0+) > 0$, the first integral in (18) is negative and exceeds $-\bar{p}\bar{h}$. The last two terms in $(\mathcal{H}V'')(a)$ are negative with sum at least $-(\rho/r)\bar{p}\bar{h}$, by the assumed bound on V'' . So $0 > (\mathcal{H}V'')(a) \geq -\beta(\bar{p}\bar{h} + (\rho/r)\bar{p}\bar{h}) = -(\rho/r)\bar{p}\bar{h}$. Since $V'(a)$ and $f(p, a)$ are both continuous in a , $\mathcal{H}: \mathcal{B} \rightarrow \mathcal{B}$.

Next, \mathcal{H} obeys the Blackwell sufficient conditions for a contraction: First, \mathcal{H} is trivially monotone in V'' . To see discounting, observe that since $V'(a) \leq V'(a-x)$ and $a < \infty$, we have $F(V'(a), \infty) + \int_0^a \int_{V'(a-x)}^\infty dF \leq 1$. Then $(\mathcal{H}V'' + b)(a) \leq (\mathcal{H}V'')(a) + \beta b$. So \mathcal{H} has a unique fixed point $\mathcal{H}V'' = V'' \in \mathcal{B}$. \square

A.6 Increasing Search Frictions: Proof of Theorem 5

Since \mathcal{T} and \mathcal{S} are monotone operators, any parametric change that increases the operator increases its fixed point. In particular, since each operator falls in r and rises in k , so do the value V and marginal value V' . Rephrasing monotonicity of V' , we see that V is strictly submodular in (a, r) and strictly supermodular in (a, k) .

Neither operator is monotone in ρ , and we instead use the method of policy improvement. Take $\rho' > \rho$ and consider a different feasible policy for ρ' : Supply $\mathcal{Y}(p, x, a|\rho)$ with

chance ρ/ρ' , and otherwise 0. The value $\tilde{V}(a|\rho')$ of this policy obeys the recursion:

$$\tilde{V}(a|\rho') = \frac{ak}{r+\rho} + \frac{\rho}{r+\rho} \mathbb{E} \left(P\mathcal{Y}(P, X, a|\rho) + \tilde{V}(a - \mathcal{Y}(P, X, a|\rho)|\rho') \right) \quad (19)$$

The unique solution for (19) is obviously $\tilde{V}(a|\rho') = V(a|\rho)$. Thus, using the *unique* optimal policy $\mathcal{Y}(p, x, a|\rho')$ yields a higher value $V(a|\rho') > V(a|\rho)$.

We show that V' increases in either parameter $\theta = -r, \rho, k$, where V' solves $V'(a) = \mathcal{S}(V', \theta)(a)$, recalling (3). By the Lemma in Albrecht, Holmlund, and Lang (1991), V' is differentiable in a parameter θ , and the continuous derivative is the unique solution to $V'_\theta(a) = \mathcal{S}_\theta(V', \theta)(a) + \mathcal{S}_{V'}(V'_\theta, V', \theta)(a)$, whose right side is a contraction map in V'_θ :

$$(\mathcal{Q}V'_\rho)(a) = (-k + r(V'(a) + \sigma'(a)))/(r + \rho)^2 + \mathcal{S}_{V'}(V'_\rho, V', \rho)(a) \quad (20)$$

$$(\mathcal{R}V'_{-r})(a) = (k + \rho(V'(a) + \sigma'(a)))/(r + \rho)^2 + \mathcal{S}_{V'}(V'_{-r}, V', -r)(a) \quad (21)$$

$$(\mathcal{K}V'_k)(a) = 1/(r + \rho) + \mathcal{S}_{V'}(V'_k, V', k)(a) \quad (22)$$

WLOG, let $\theta = \rho$, and focus on the \mathcal{Q} recursion. If $V'_\rho \geq 0$ then $\mathcal{S}_{V'}(V'_\rho, V', \rho)(a) \geq 0$, and so the second term in (20) is nonnegative. Since $\sigma'(a) > 0$ by Corollary 3, we have $(r + \rho)^2(\mathcal{Q}V'_\rho)(a) > -k + rV'(a)$. Hence, $(\mathcal{Q}V'_\rho)(a) > 0$ as $V'(a) > k/r$ by Lemma 2. In light of the second conclusion of Corollary 3.2.1 in Lucas, Stokey, and Prescott (1989), the fixed point obeys $\mathcal{Q}V'_\rho = V'_\rho > 0$, and thus V is strictly supermodular in (a, ρ) .

Next, we argue recursively from (20)–(22) that $V'_\theta(a)$ falls in a for $\theta = \{-r, \rho, k\}$. By the strict concavity of V , it suffices to show that $\mathcal{S}_{V'}(V'_\theta, V', \theta)(a)$ falls in a . Differentiate (7) in θ , and change the order of integration to integrate with respect to $z = a - x$:

$$\mathcal{S}_{V'}(\theta, V'; V'_\theta)(a) = \frac{\rho}{r + \rho} \left(V'_\theta(a)F(V'(a), \infty) + \int_0^\infty \int_{a-\mathcal{Y}(p, a, a)}^\infty V'_\theta(z)f(p, a - z)dzdp \right) \quad (23)$$

The first term in (23) falls in a by the concavity of V , and the second since, by Theorem 3, $a - \mathcal{Y}(p, a, a) = \min\{a, (V')^{-1}(\min\{p, V'(0+)\})\}$ increases in a , and the density $f(p, a - z)$ falls in a . As a result, $\mathcal{S}V'_\theta = V'_\theta$ falls in a for $\theta = \{-r, \rho, k\}$. \square

A.7 Waiting Times: Proof of Theorem 7 Finished

Define the hazard rate $\varphi(p) = F_1(p, \infty)/\bar{F}(p, 0)$. As V' is continuously differentiable in ρ (see §A.6), the elasticity $\mathcal{E}_\rho(\Phi(a)) = -\varphi(V'(a))V'_\rho(a)\rho$ is well-defined. Now, $V(a)$ is concave by Theorem 2 and $V'(a)$ is submodular in (a, ρ) by Theorem 5. Also, $\varphi(p)$ is

increasing since $g(p)$ is log concave, and thus so too is $\bar{F}(p, 0)$. So the absolute elasticity $|\mathcal{E}_\rho(\Phi(a))|$ falls in a . So $|\mathcal{E}_\rho(\Phi(a))| \leq |\mathcal{E}_\rho(\Phi(0+))| = \varphi(V'(0+))V'_\rho(0+)\rho$. From (20):

$$(r + \rho)V'_\rho(0+) = \frac{-k}{r + \rho} + \frac{r}{r + \rho}\mathbb{E}(\max\{P, V'(0+)\}) + \rho V'_\rho(0+)F(V'(0+), \infty) \quad (24)$$

Now, (3) implies $\mathbb{E}(\max\{P, V'(0+)\}) = ((r + \rho)V'(0+) - k)/\rho$. Substituting into (24):

$$\rho V'_\rho(0+) = \frac{r(V'(0+) - k/r)}{r + \rho\bar{F}(V'(0+), 0)} < V'(0+) - k/r \quad (25)$$

Since $rV'(0+) = k + \rho\mathbb{E}[\max\{P - V'(0+), 0\}]$, recalling our parallel wage search problem, the right side of (25) is less than $\rho\mathbb{E}[P]/r$. Altogether, $0 \leq |\mathcal{E}_\rho(\Phi(a))| \leq \varphi(V'(0+))\rho\mathbb{E}[P]/r$. So, $|\mathcal{E}_\rho(\Phi(a))| \rightarrow 0$ as $\rho \rightarrow 0$ (since $V'(0+) \rightarrow k/r$). The sandwich inequality also implies $|\mathcal{E}_\rho(\Phi(a))| \leq 1$ if ρ is low enough, say $\rho \leq \bar{\rho}$. Finally, $\mathcal{E}_\rho(\xi(a)) = 2\mathcal{E}_\rho(\tau(a)) = 2(-1 + |\mathcal{E}_\rho(\Phi(a))|) \leq 0$ if ρ is low enough. \square

A.8 Transactional Liquidity Measures: Proof of Theorem 8

As the seller's objective function (1) in a trade $py + V(a - y)$ is supermodular in (y, θ) , for $\theta = r, -\rho, -k$ by Theorem 5, $\mathcal{Y}(p, x, a)$ rises in θ (Theorem 6.1 in Topkis (1978)). By the same logic, $py + V(a - y) - \chi_{[0, a](y)}$ is supermodular in (y, a) , so $\mathcal{Y}(p, x, a)$ rises in a .

Since $V'(a)$ is decreasing and convex by Theorem 4, $\pi(y, a)$ is increasing in y , and decreasing in a . Likewise, $\pi(y, a)$ falls in $\theta = r, -\rho, -k$, since $V'(a)$ is supermodular in each pair (a, θ) , by Theorem 5.

Supply is constant in p and equal to $\min\{x, a\}$ if $p \geq V'(0+)$ or constant in p and zero if $p \leq V'(a)$. Otherwise, the supply is $Y(p, a) = a - (V')^{-1}(p)$ and thus $\eta(p, a) = Y_1(p, a)p/Y(p, a)$. Since the supply is additively separable in (a, p) , its slope is constant in a . Then $\eta(p, a)$ is decreasing and convex in a , since $p/Y(p, a) = p/(a - (V')^{-1}(p))$ is decreasing and convex in a . Clearly as $a \rightarrow \infty$, $\eta(p, a) \rightarrow 0$.

Finally, $\lambda(y, a) = -1/V''(a - y)$. This is decreasing in a and increasing in y by Theorem 4. It rises in θ since $V''(a)$ rises in θ by Theorem 5. \square

We finally prove a claim in footnote 8 that the sales rate $\rho\mathbb{E}[\mathcal{Y}(P, X, a)]$ rises in ρ , for low ρ . By Theorem 4, the supply $Y(p, a)$ is differentiable in the domain $\mathcal{D}(x, a) = \{p | V'(a) \leq p \leq V'(a - \min\{x, a\})\}$. The sales rate derivative is

$$\rho\partial\mathbb{E}[Y(p, a)]/\partial\rho + \mathbb{E}[\mathcal{Y}(P, X, a)] = \rho(\mathbb{E}[Y_\rho(P, a)\mathbb{1}_{P \in \mathcal{D}(X, a)}] + \mathbb{E}[\mathcal{Y}(P, X, a)]/\rho) \quad (26)$$

As $\rho \rightarrow 0$, we have $V'(a) \rightarrow k/r$ by Lemma 3, whence the domain $\mathcal{D}(X, a)$ collapses to a

point. To wit, $\mathbb{E}[Y_\rho(P, a)\mathbb{1}_{P \in \mathcal{D}(X, a)}] \rightarrow 0$. The term $\mathbb{E}[\mathcal{Y}(P, X, a)]/\rho$ falls from ∞ to 0 as ρ increases. So for small ρ , the right side of (26) is positive, i.e. $\partial(\rho\mathbb{E}[\mathcal{Y}(P, X, a)])/\partial\rho \geq 0$.

A.9 Liquidity with Bargaining

By the identity $\mathcal{P}(w, x, a) \equiv p(\Upsilon(w, a), a)$, the bargained supply elasticity $\mathcal{H}(\mathcal{P}(w, x, a), a)$ solves:

$$\frac{1}{\mathcal{H}(\mathcal{P}(w, x, a), a)} = \frac{1}{\mathcal{P}(w, x, a)} \left(\frac{\delta w}{\mathcal{E}_w(\Upsilon(w, a))} + (1 - \delta)(w - c(\Upsilon(w, a), a)) \right) \quad (27)$$

We have shown (in §6, point 4) that $c(\Upsilon(w, a), a)$ and $\mathcal{P}(w, x, a)$ fall in a , and Theorem 8 proves that the elasticity $\mathcal{E}_w(\Upsilon(w, x, a))$ falls in a . The right side of (27) rises in a . So the bargained supply elasticity $\mathcal{H}(\mathcal{P}(w, x, a), a)$ falls in the position a .

Next, since the two other transactional liquidity measures — depth and purchase premium — are expressed in terms of the inverse uncapped supply (14), it suffices to understand the marginal value \mathcal{V}' and secant slope $c(y, a)$.

Lemma A.1 *The secant slope $c(y, a)$ is increasing in y , ρ , k , and δ , falling in a and r , supermodular in $(y, -a)$, $(y, -r)$, (y, ρ) , (y, δ) , and (y, k) , and convex in y .*

Proof: By the earlier equivalence result for the bargaining model, all monotonicity claims in Theorems 4–5 are inherited by \mathcal{V}' , and thus by $\int_0^y \mathcal{V}'(a - t)dt/y$. By the same logic, supermodularity claims about $c(y, a)$ follow from monotonicity of $c_1(y, a) = -y^{-2} \int_0^y \int_{a-y}^{a-t} \mathcal{V}''(u)du dt$. For the convexity in y , change the order of integration, and change variables, to get $c_1(y, a) = -\int_0^1 \mathcal{V}''(a - zy)zdz$. This rises in y by Theorem 4. \square

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