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Rationalizability and Mixed Strategies in Large Games*

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Abstract

We show that in large games with a finite set of actions in which the payoff of a player depends only on her own action and on an aggregate value that we call the *state* of the game, which is obtained from the complete action profile, it is possible to define and characterize the sets of (Point-)Rationalizable States in terms of pure and mixed strategies. We prove that the (Point-)Rationalizable States sets associated to pure strategies are equal to the sets of (Point-)Rationalizable States associated to mixed strategies. By example we show that, in general, the Point-Rationalizable States sets differ from the Rationalizable States sets.

KEYWORDS: Rationalizable Strategies, Large Games, Non-atomic Games, Expectational Coordination, Strong Rationality.

JEL CLASSIFICATION: D84, C72, C62.

1 Introduction

Rationalizability has been used to give an eductive justification for the rational expectations hypothesis since the works of [Guesnerie \(1992, 2002\)](#), where he defines a Strongly Rational Equilibrium as an equilibrium that is the only “rationalizable outcome” of an economic model. The concept of rationalizability had been defined and studied only in games with a finite number of players, while the applications to expectational coordination were usually in the context of models with a continuum of agents.

Rationalizability is formally defined and characterized in [Jara-Moroni \(2012\)](#) for games with a continuum of players, compact strategy sets and continuous payoff functions. The model of choice was the one presented in [Rath \(1992\)](#), where the payoff of each player depends on her action and the average action of all the players. The rationalizable sets are proved to be the result of the iterative elimination of non-best reply average actions, extending the results of [Bernheim \(1984\)](#), [Pearce \(1984\)](#) and [Tan and da Costa Werlang \(1988\)](#) to this class of large games.

The incipient literature on large games that builds over [Rath’s](#) and [Schmeidler’s](#) model usually assumes that the set of actions in a large game is finite or the unit simplex in \mathbb{R}^n (which can be interpreted as the set of mixed strategies of a finite action game) ([Rath, 1994, 1995, 1998](#); [Carmona, 2004](#); [Khan et al., 2006](#), among others).

In this note we study rationalizability in the context of games with a continuum of players from the point of view of finite action sets and the set of mixed strategies of a finite action set game. We claim that regarding rationalizability it is superfluous for the modeler to consider mixed strategies since the expectational implications of this consideration are already obtained with the (slightly) less involved model of pure

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strategies. In words, in terms of forecasting forecasts of the rivals, players do not care whether a fraction of players chose one action and the rest chose another action, or whether every player chooses both actions with certain probabilities that provide the same average action.

The remainder of the paper is as follows: in Section 2 we introduce games with a continuum of players and an average variable that we call the *state* of the game, and following Jara-Moroni (2012) we define and characterize the sets of (Point-)Rationalizable States; in Section 3 we claim and prove our main result and we close with some remarks in Section 4.

2 Rationalizability in games with a continuum of players

Let I , the unit interval in \mathbb{R} endowed with the Lebesgue measure λ , be the *set of players*. All the players have the same *set of available actions* A a compact subset of \mathbb{R}^n . A *strategy profile* is a measurable function from I to A and the set of strategy profiles is denoted by A^I . Let S be the set of all integrals of strategy profiles $S := \left\{ \int_I \mathbf{a} \, d\lambda \mid \mathbf{a} \in A^I \right\}$. Then S is the convex hull of A and so is also compact. The elements of S are the *aggregate states* and S is the *set of states*. We denote the set of real-valued bounded continuous functions defined on a space X by $C_b(X)$. Let $\mathcal{U}_{A \times S}$ denote the topological space of $C_b(A \times S)$ endowed with the sup norm topology. A *game with a continuum of players* is a measurable function $\mathbf{u} : I \rightarrow \mathcal{U}_{A \times S}$ that associates players with their payoff functions. A *Nash Equilibrium* is a strategy profile $\mathbf{a}^* \in A^I$ such that λ -almost-everywhere in I :

$$u\left(i, \mathbf{a}^*(i), \int_I \mathbf{a}^*(i) \, di\right) \geq u\left(i, y, \int_I \mathbf{a}^*(i) \, di\right) \quad \forall y \in A.$$

This formulation is Rath's (1992) extension of Schmeidler's (1973) formulation of games with a continuum of players. To denote *games with a continuum of players* that have an *aggregate state* as above, we will use the notation \mathbf{u} . We will denote then equivalently $\mathbf{u}(i)$ and $u(i, \cdot, \cdot)$. In this framework Rath (1992) shows that for every game there exists a pure strategy Nash Equilibrium.

2.1 (Point-)Rationalizable States

An aggregate state is rationalizable if it is consistent with the assumptions of rationality and common certainty of rationality. Following Bernheim (1984), Pearce (1984) and Guesnerie (1992), Jara-Moroni (2012) has defined and characterized rationalizable states, as the result of the iterative elimination of states that are not consequence of rational behavior of players. Given the state set S , all players must believe that each player will only play an action that is a best response to some forecast over states. Actual action profiles then must be selections of a best-response correspondence constructed, for each player, as the image through their best-response mapping of the complete set of forecasts over S . Then, players should believe that the only states that may rise as outcomes are elements of the subset of S that results from taking the integrals of these selections. This same reasoning may be applied to this new set of states. The process may continue ad infinitum.

Following the terminology of Bernheim we will distinguish between *rationalizability* and *point-rationalizability*. Rationalizability differs from point-rationalizability in that forecasts are probability distributions whose supports are contained in the sets of outcomes; while for point-rationalizability, forecasts are points or Dirac probability measures on these sets.

For each player $i \in I$ consider the set-valued mapping $\rho(i, \cdot) : \Delta(S) \rightrightarrows A$ that gives the actions that maximize expected utility given a probability measure μ over the set of states S , $\mu \in \Delta(S)$:

$$\begin{aligned} \rho(i, \mu) &:= \operatorname{argmax}_{y \in A} \mathbb{E}_\mu [u(i, y, s)] \\ &= \operatorname{argmax}_{y \in A} \int_S u(i, y, s) \, d\mu(s). \end{aligned}$$

If player i holds the forecast μ , then rational behavior implies that she will chose an action from the set $\rho(i, \mu)$. If μ is an atomic measure concentrated on a point s , we will use the notation $\hat{\rho}(i, s)$. This is,

$\hat{\rho}(i, s) := \rho(i, \delta_s)$. It is seen directly that for a given action profile \mathbf{a} , in the context of a game \mathbf{u} , the best response of a player i to \mathbf{a} is given by $\hat{\rho}(i, \int \mathbf{a})$.

Denoting by $\mathcal{B}(X)$ the family of Borel subsets of X , we define the mapping $R : \mathcal{B}(S) \rightarrow \mathcal{B}(S)$ that - to each Borel set $X \subseteq S$ - associates the set $R(X) \subseteq S$ defined by:¹

$$R(X) := \int_I \rho(i, \Delta(X)) di, \quad (2.1)$$

where $\rho(i, \Delta(X))$ is the image through the best-reply mapping of the set of probability measures with support in X . If forecasts are constrained to $\Delta(X)$, each player $i \in I$ will use an action in $\rho(i, \Delta(X))$ and so the state of the game will be the integral of an action profile selected from $i \Rightarrow \rho(i, \Delta(X))$ and thus an element of $R(X)$.

We say that a set $X \subseteq S$ satisfies the *best-reply property* if it is a subset of its image through R , this is:

$$X \subseteq R(X) \quad (2.2)$$

A set that satisfies the best-reply property contains only states that result from optimal reactions to forecasts constrained to this same set.

Definition 2.1. The set of Rationalizable States is the maximal subset $X \subseteq S$ that satisfies:

$$X \subseteq R(X)$$

and we denote it \mathbb{R}_S .

The set-to-set mapping R allows us to rigorously define the process of iterative elimination of non-best-reply states that characterizes the set of rationalizable states.

$$R^0(S) := S \quad \text{and} \quad R^{t+1}(S) := R(R^t(S)) \quad \text{for } t \geq 0. \quad (2.3)$$

$$\mathbb{R}'_S := \bigcap_{t=0}^{\infty} R^t(S) \quad (2.4)$$

Theorem 2.2 (Jara-Moroni (2012)). *The set of Rationalizable States of a game \mathbf{u} can be calculated as*

$$\mathbb{R}_S = \mathbb{R}'_S$$

This same exercise can be developed to define and characterize Point-Rationalizable States. For this, simply consider $\hat{\rho}$ instead of ρ in (2.1). Thus, denoting by 2^X the family of subsets of X we define the mapping $\hat{R} : 2^S \rightarrow 2^S$

$$\hat{R}(X) := \int_I \hat{\rho}(i, X) di. \quad (2.5)$$

The set $\hat{R}(X)$ gives the set of states that are obtained as a consequence of optimal behavior when forecasts are deterministic and constrained to the set X . Since any Borel set $X \subseteq S$ can be embedded into $\Delta(X)$, the inclusion $\hat{\rho}(i, X) \subseteq \rho(i, \Delta(X))$ holds and so we have that $\hat{R}(X) \subseteq R(X)$. The definition and characterization of Point-rationalizable states are analogous to Definition 2.1 and Theorem 2.2.

¹The integral of a correspondence $F : I \Rightarrow \mathbb{R}^n$ is calculated, following Aumann (1965), as the set of integrals of all the integrable selections of F . This is,

$$\int_I F(i) di = \left\{ \int_I f(i) di : f \text{ is an integrable selection of } F \right\}$$

where $\int f di := (\int f_1(i) di, \dots, \int f_n(i) di)$.

Definition 2.3. The set of Point-Rationalizable States, \mathbb{P}_S , is the maximal subset $X \subseteq S$ that satisfies

$$X \subseteq \hat{\mathbf{R}}(X). \quad (2.6)$$

Theorem 2.4 (Jara-Moroni (2012)). *The set of Point-Rationalizable States of a game \mathbf{u} can be calculated as*

$$\mathbb{P}_S = \mathbb{P}'_S = \bigcap_{t=0}^{\infty} \hat{\mathbf{R}}^t(S) \quad (2.7)$$

3 Finite action set and mixed strategies

We are interested in games in which the set of actions is finite and we will consider mixed strategies. The fundamental for our model will be a game \mathbf{u} as defined in Section 2 where the set of actions (or pure strategies) is $A = \{a^1, \dots, a^n\}$, where a^k is the k th unit vector of \mathbb{R}^n . The set of mixed strategies, denoted by M , is the set of probability measures over A and so is the unit simplex in \mathbb{R}^n (which we denote by Δ). A *mixed strategy profile* is a measurable function from I to M and the set of mixed strategy profiles is denoted by M^I . Let S^M be the set of all integrals of mixed strategy profiles $S^M := \{\int_I \mathbf{m} \, d\lambda \mid \mathbf{m} \in M^I\}$. Then S^M is as well the simplex in \mathbb{R}^n , as is the set of all integrals of action profiles. This is, the simplex in \mathbb{R}^n stands for the set of mixed strategies, the set of aggregate states due to pure strategies and the set of aggregate states due to mixed strategies. Given \mathbf{u} , for each player $i \in I$ we calculate payoff associated to a profile of mixed strategies with the function $U(i, \cdot) : \Delta \times \Delta \rightarrow \mathbb{R}$ as follows:

$$U(i, m(i), s) := \sum_{k=1}^n m^k(i) \cdot u(i, a^k, s); \quad (3.1)$$

so that,

$$U\left(i, m(i), \int_I \mathbf{m} \, d\lambda\right) = \sum_{k=1}^n m^k(i) \cdot u\left(i, a^k, \int_I \mathbf{m} \, d\lambda\right), \quad (3.2)$$

where $m^k(i)$ is the probability that player i gives to action k in her mixed strategy.²

Note that starting from \mathbf{u} we have defined its mixed extension \mathbf{U} which is a measurable function from I to $\mathcal{U}_{\Delta \times \Delta}$ and so is in its own terms a game with a continuum of players as defined in Section 2 (with $A = S = \Delta$). We may then consider four rationalizable sets, the two Rationalizable States sets associated to pure strategies \mathbb{R}_S and \mathbb{P}_S ; and two new sets, the sets of (Point-)Rationalizable States associated to mixed strategies, which we will denote by \mathbb{R}_S^M and \mathbb{P}_S^M .

Theorem 3.1. *In a game with a continuum of players and finite set of actions the sets of (Point-)Rationalizable States associated to pure and mixed strategies are equal. Formally:*

$$\mathbb{R}_S = \mathbb{R}_S^M \qquad \mathbb{P}_S = \mathbb{P}_S^M.$$

Proof. The proof proceeds by demonstrating that the process of elimination of states for both (Point-)Rationalizable States sets (the one associated to mixed strategies and the one associated to pure strategies) are the same.

We need to introduce some notation: we will denote with a superscript M the mathematical objects related to the process associated to mixed strategies and without superscript the ones associated to pure strategies (or actions). Thus, for instance \mathbf{R} represents the process of elimination of states when considering only actions and \mathbf{R}^M represents the process considering mixed strategies.

²Since the functions $\mathbf{u}(i)$ belong to $\mathcal{U}_{A \times S}$ the function U is well defined.

It is clear that for any Borel set $X \subseteq \Delta$, $R(X) \subseteq R^M(X)$, since pure strategies are as well mixed strategies.

From (2.1) we see that for a Borel set $X \subseteq \Delta$, $R^M(X)$ is the integral over players of the image through the best-reply mapping of the set of probability measures with support in X (the set of general forecasts constrained by X). For a given forecast $\mu \in \Delta(X)$ the expected utility in mixed strategies is:

$$\mathbb{E}_\mu [U(i, m(i), s)] = \mathbb{E}_\mu \left[\sum_{k=1}^n m^k(i) \cdot u(i, a^k, s) \right] \quad (3.3)$$

$$= \sum_{k=1}^n m^k(i) \cdot \mathbb{E}_\mu [u(i, a^k, s)]. \quad (3.4)$$

So mixed strategies that maximize expected payoff (expectation that comes from the forecast) are those that put positive probability only in the actions that maximize expected payoff with respect to the same forecast. Thus, for a given forecast μ we can construct the set of maximizers in mixed strategies from the set of maximizers in pure strategies as follows:

$$\rho^M(i, \mu) = \text{co} \{ \rho(i, \mu) \}. \quad (3.5)$$

Now take a state $s \in R^M(X)$. This means that $s = \int_I m(i) di$ where \mathbf{m} is a measurable selection of the set valued mapping (s.v.m.) $i \rightrightarrows \rho^M(i, \Delta(X))$. This implies that for every player $i \in I$ there exists a forecast $\mu(i) \in \Delta(X)$ such that $m(i) \in \rho^M(i, \mu(i))$. This is, \mathbf{m} is a measurable function and is a selection from the s.v.m. $i \rightrightarrows \rho^M(i, \mu(i))$. This immediately implies that the integral of $i \rightrightarrows \rho^M(i, \mu(i))$ is non-empty (since s is an element of this set). From (3.5) we get that \mathbf{m} is a measurable selection of $i \rightrightarrows \text{co} \{ \rho(i, \mu(i)) \}$ and so $s \in \int_I \text{co} \{ \rho(i, \mu(i)) \} di$. From Aumann (1965) we know that:³

$$\int_I \text{co} \{ \rho(i, \mu(i)) \} di = \int_I \rho(i, \mu(i)) di;$$

and so $s \in \int_I \rho(i, \mu(i)) di$. Evidently, $\rho(i, \mu(i)) \subseteq \rho(i, \Delta(X))$ so:

$$s \in \int_I \rho(i, \Delta(X)) di = R(X).$$

We conclude that $R^M(X) \subseteq R(X)$.

Since for any Borel set $X \subseteq S$, $R^M(X) = R(X)$ we get that the two processes of elimination of states coincide and so $\mathbb{R}_S = \mathbb{R}_S^M$.

For the second equality, take $s \in \hat{R}^M(X)$ and simply consider the subset $X \subseteq \Delta(X)$ instead of $\Delta(X)$ in the previous reasoning to conclude that $\hat{R}^M(X) \subseteq \hat{R}(X)$. ■

The following example, taken from Jara-Moroni (2012), shows that the sets \mathbb{R}_S and \mathbb{P}_S may differ.

Example 1. Consider the game where

$$A = \{a^1, a^2, a^3\}, \quad S = \Delta = \{(s_1, s_2, s_3) \in \mathbb{R}_+^3 \mid s_1 + s_2 + s_3 = 1\}$$

and all the players have the same payoff function $\mathbf{u}(i) = u : A \times \Delta \rightarrow \mathbb{R}$ defined by:

$$u(a^1, s) = \begin{cases} 0 & \text{if } s_3 \leq 1/3 \\ 18s_3 - 6 & \text{if } \frac{1}{3} < s_3 < \frac{1}{2} \\ 3 & \text{if } s_3 \geq 1/2 \end{cases}; \quad u(a^2, s) = 2; \quad \text{and} \quad u(a^3, s) = \begin{cases} 3 & \text{if } s_3 \leq 1/2 \\ 12 - 18s_3 & \text{if } \frac{1}{2} < s_3 < \frac{2}{3} \\ 0 & \text{if } s_3 \geq 2/3 \end{cases}.$$

³The integral of a s.v.m. $t \rightrightarrows F(t)$ and the integral of the s.v.m. $t \rightrightarrows \text{co} \{F(t)\}$ coincide.

We see that both the original game \mathbf{u} and its extension \mathbf{U} are games with a continuum of players.

Clearly, the action a^2 is never a best response to deterministic forecasts, since for any $s \in \Delta$ either a^1 or a^3 provide payoff equal to 3. Thus, $\hat{\rho}(\Delta) = \{a^1, a^3\}$ and so $\hat{\mathbb{R}}(\Delta) = \text{co}\{a^1, a^3\}$. Moreover $\hat{\rho}(\text{co}\{a^1, a^3\}) = \{a^1, a^3\}$, so $\hat{\mathbb{R}}^2(\Delta) = \text{co}\{a^1, a^3\}$. We conclude that $\mathbb{P}_S = \text{co}\{a^1, a^3\}$.

However, if a forecast $\mu \in \Delta(S)$ is such that $\mu(\{s \in \Delta : s_3 = 0\}) = \mu(\{s \in \Delta : s_3 = 1\}) = 1/2$ we have that,

$$U(a, \mu) = \begin{cases} 1.5 & \text{if } a = a^1 \\ 2 & \text{if } a = a^2 \\ 1.5 & \text{if } a = a^3 \end{cases}$$

and so $\rho(\mu) = a^2$, so $\mathbb{R}_S = \Delta$.

In this example then, the rationalizable sets are well defined and differ.

4 Concluding remarks

We have seen that in the version of the model of games with a continuum of players due to [Schmeidler \(1973\)](#) and [Rath \(1992\)](#) where payoff depends on the average of the actions of all the players, it is possible to build the mixed extension of the a game with a continuum of players and finite (pure) strategy sets. This construction is in its own right a game with a continuum of players.

Consequently the theory and properties of (Point-)Rationalizable States introduced and studied in [Jara-Moroni \(2012\)](#) are applicable to these two settings. Taking advantage of the properties of integrals of set valued mappings we have been able to prove that the sets of (Point-)Rationalizable States associated to mixed and pure strategies are equal. We see then than in games with finite sets of actions, in terms of rationalizability, it is irrelevant to consider mixed strategies, so a modeler that wishes to study expectational coordination may simply consider pure strategies. This result is a contribution to the literature on purification of equilibria ([Khan et al., 1997, 2006](#); [Balder, 2008](#)). However, [Example 1](#) shows that the choice of forecasts, even in games with finite sets of actions, is not superfluous and may in general alter the conclusions depending on whether the modeler is considering general or deterministic forecasts. Nevertheless, if the game in consideration is affine in the state variable then the type of forecasts are superfluous ([Jara-Moroni, 2012](#)) and so the modeler may well work with pure strategies and deterministic forecasts.

A question that remains unanswered is the comparison between the set of equilibrium states and the set of (point-)rationalizable states. It is shown in [Jara-Moroni \(2012\)](#) that the convex hull of the set of equilibrium states is contained in the rationalizable sets and this inclusion can be proper (illustrative examples may be found in [Rath \(1994, 1995, 1998\)](#)). A natural question is whether one can provide conditions for these sets to be identical. One may conjecture that in games with finite number of actions, if the set of equilibrium states is contained in the set of actions, then the sets are identical. However, we may provide a simple game with two actions and constant utility function $i \rightarrow u(i, \cdot, \cdot)$ where this property does not hold. Nevertheless, it is important for the conjecture to fail that at equilibrium, the equilibrium action was not the unique best response. Thus, we believe that it is possible to prove that if in addition equilibria are strict (each player has a unique best response), then the property follows. An interesting case to explore as well is that of Nice Games with a continuum of players.

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