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Strongly Rational Equilibrium in a Global Game with Strategic Substitutes

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Strongly Rational Equilibrium in a Global Game with Strategic Substitutes*

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Abstract

The Global Games literature emerged as a device aimed at equilibrium selection in games (Carlsson & Van Damme, 1993). Uniqueness selection of equilibrium has been proved for a general setting, but restricted to models with supermodular payoffs (Frankel, Morris & Pauzner, 2002). Moreover, this unique equilibrium profile is obtained through iterated deletion of strictly dominated strategies and so it is the only rationalizable outcome of the game. This implies, following Guesnerie (1992) that it is as well a strongly rational equilibrium. For the case of Global Games with strategic substitutes uniqueness of equilibrium has been proved in Harrison (2003), but iterative elimination of strictly dominated strategies does not necessarily deliver this equilibrium. This allows for multiple rationalizable strategies, preventing to conclude about strong rationality. Motivated by Guesnerie & Jara-Moroni (2011) in this work study a simple global game with strategic substitutes and provide conditions for the existence of a strongly rational equilibrium. This opens an unexplored research agenda on the study of strongly rational equilibria in global games with strategic substitutes.

1 Introduction

Global Games are games of incomplete information, where the players' payoffs depend on an uncertain state that represents the fundamental of the modeled situation, from which each player receives a (potentially different) signal with a small amount of noise. In these games, the noise technology is common knowledge so each players' signal generates beliefs about fundamentals of the model and the other players' beliefs (over fundamentals and beliefs of their rivals and so on). Originally, global games where assessed as equilibrium selection devices and, in time, they have become as well a useful methodology to simplify the analysis of high-order beliefs in strategic settings. Our interest relates to their equilibrium selection application.

Global Games where first introduced by Carlsson and van Damme (1993) as a means to depart from the assumption that players are excessively rational and well-informed with respect to the real-life situation in scrutiny. The idea behind this equilibrium selection

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approach is to examine the set of Nash equilibria of a game as a limit of equilibria of payoff-perturbed games and observe any reduction in the set. For a given realization of the state and its associated complete information game, the global game approach may allow to select a unique equilibrium in this game, provided that there is a unique equilibrium in the incomplete information game that results when the noise in the players' observation is sufficiently small.

Carlsson and van Damme (1993) show that for a general class of two-player, two-action games, this limit comprises a single equilibrium profile. Moreover, the equilibrium profile is obtained through iterated deletion of strictly dominated strategies. Roughly, the deletion requires that, for each player and for each action of that player, there are certain extreme values of the state, for which that action is strictly dominant. Even if these values carry very little probability weight, the players can use signals close to these "dominance regions" to rule out certain types of behavior of others. Hence, the iterative deletion proceeds.

These results have been extended by Frankel et al. (2003) to general finite games[1] with real valued set of actions. However, existing results in this literature are typically limited to the case of strategic complementarities (and some other technical assumptions). This strong result is very useful for many games such as bank run models, currency crises and some type of herding behavior among others.[2]

Global games with strategic substitutes had not been as thoroughly studied as the case of strategic complements. Uniqueness of equilibrium can not be obtained by simply passing from the strategic complements model of Frankel et al. (2003) to the strategic substitutes environment. The elimination of strictly dominated strategies does not provide a unique outcome and so this technique may not be used to prove uniqueness of equilibrium. However, adding a minimum of player heterogeneity Harrison (2003) showed that the equilibrium is unique in a fairly general model with strategic substitutes. Still, this unique equilibrium may not be the unique outcome of the iterative elimination of strictly dominated strategies (Morris, 2009).

Obtaining a unique equilibrium as the result of the elimination of strictly dominated strategies implies that this equilibrium is *strongly rational*. The concept of strongly rational equilibrium was first stated by Guesnerie (1992) as a mean to provide an eductive foundation for the rational expectations hypothesis. Following Guesnerie (1992, 2002) an equilibrium is strongly rational, if it is the only *rationalizable strategy profile* of a game. The uniqueness of the rationalizable solution depends solely on the fundamentals of the model. Strong rationality has been studied in the context of complete information in terms of stability of equilibria by Evans and Guesnerie (1993, 2003, 2005), Chamley (1999, 2004) and Desgranges and Heinemann (2005). Morris and Shin (1998) incorporated rationalizability under incomplete information to the global games literature studing stability properties of equilibria (Morris and Shin, 2003) As in global games, optimistic stability results, in terms of strong rationality, have been obtained in the context of models that present strategic complements. Namely, a unique equilibrium is strongly rational under strategic complementarity. Such positive results are harder to obtain in environments with strategic substitutes. Guesnerie and Jara-Moroni (2011) find conditions for strongly rational equilibrium in models with a

[1] Games with a finite number of players and for each player a finite number of available actions

[2] For a complete survey of the global games literature see Morris and Shin (2003)

continuum of agents and strategic substitutes under complete information, that pertain to uniqueness of fixed points of the second iterate of a best response mapping, while Morris and Shin (2009) have attempted to link these results to global games with strategic substitutes (see as well Morris and Shin, 2005).

Although we have a result of unique equilibrium in global games with strategic substitutes, the conditions under which this result is obtained are not enough to state strong rationality of this equilibrium (Harrison, 2003). However, in the light of the results in Guesnerie and Jara-Moroni (2011), further requirements should allow to state that this unique equilibrium is in fact strongly rational. In this work we study a simple global game with strategic substitutes with heterogenous players that satisfies the conditions for uniqueness of equilibrium of the theorem in Harrison (2003). We show that with sufficient players' heterogeneity, indeed the unique equilibrium is as well strongly rational. This results solves a puzzle in the global games literature and replicates the results found in Guesnerie and Jara-Moroni (2011), regarding the passage from strategic complements to strategic substitutes. It is indeed possible to obtain a unique strongly rational equilibrium under strategic substitutes, but uniqueness of equilibrium is not sufficient as in the case of strategic complements. Additional conditions must be required.

2 Example and Motivation

Consider the following two player parameterized normal form game. The utility functions for player 1 and 2 are respectively:

$$u_1(a,x) = a_1(2 + x - (a_1 + a_2)) \qquad u_2(a,x) = a_2(2 + x - (a_1 + a_2))$$

with $a_i \in \{0,1\}$ and $x \in \mathbb{R}$ being a parameter value. A matrix representation of this game is shown in Figure 1.

		Player 2	
		$a_2 = 1$	$a_2 = 0$
Player 1	$a_1 = 1$	$x,\ x$	$1 + x, 0$
	$a_1 = 0$	$0, 1 + x$	$0, 0$

Figure 1: Matrix representation of the game.

It is easy to check that, depending on the value of x the game has different Nash equilibria sets:

- If $x > 0$ there is a unique equilibrium profile, $(1,1)$, where for each player to choose action 1 is strictly dominant.
- If $x < -1$ there is a unique equilibrium profile, $(0,0)$, where for each player to choose action 0 is strictly dominant.
- if $x \in]-1, 0[$ there are two equilibria profiles: $(0,1)$ and $(1,0)$.

We can schematically see this equilibrium behavior in Figure 2. Following the global games literature, we will refer as dominance regions to the set of values of x where the equilibrium is played with strictly dominant strategies.

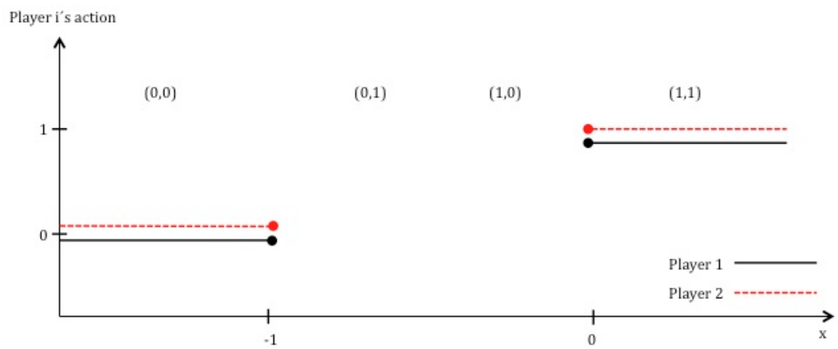

Figure 2: Nash equilibria set depending on x.

Clearly there is no ambiguity in the behavior prediction for values of x in the dominance regions of this game, but outside these regions we are unable to be equally precise. Carlsson and van Damme (1993) would select an equilibrium by passing to an incomplete information game where players are noisily informed about the value of x. If we have a unique equilibrium in the incomplete information game, as the noise goes to zero, we could use this Bayesian equilibrium strategy as a selecting device: for each $x \in]-1, 0[$ we could predict the action played by each player by evaluating her equilibrium strategy in x.

The above example has the Global Games structure, but it is not possible to apply the Carlsson and van Damme methodology because the symmetric equilibria that characterize each dominance region ($(0,0)$ and $(1,1)$), do not allow to start any process of iterated elimination of strategies in the incomplete information game.[3] Harrison (2003) made an important observation, if some ex ante asymmetry (players heterogeneity) is introduced, it is possible to create new (good) dominance regions. We can do this in our example by introducing a "cost" ξ to player 2 in her utility function.

$$u_1(a,x) = a_1 \left(2 + x - (a_1 + a_2)\right) \qquad u_2(a,x) = a_2 \left(2 + x - \xi - (a_1 + a_2)\right)$$

This defines a new game with different sets of equilibria. Figure 3 shows how these sets depend on the value of x, and that new dominance regions are introduced.

Considering the incomplete information version of this game, we can now use these dominance regions to start the process of iterated deletion of strictly dominated strategies to select an equilibrium. Figure 4 characterizes the selected equilibrium, in which each player uses a switching strategy from 0, to 1 with different cutoff values. Player 1 switches in $x = -1$ and player 2 in $x = \xi$.

Harrison (2003) shows that uniqueness of equilibrium in the incomplete information game, with sufficiently small noise, can be generalized to a class of global games with strategic

[3] Moreover, the Carlsson and van Damme result states that the selected equilibrium must be risk dominant, but in our example neither of the two equilibria has this property.

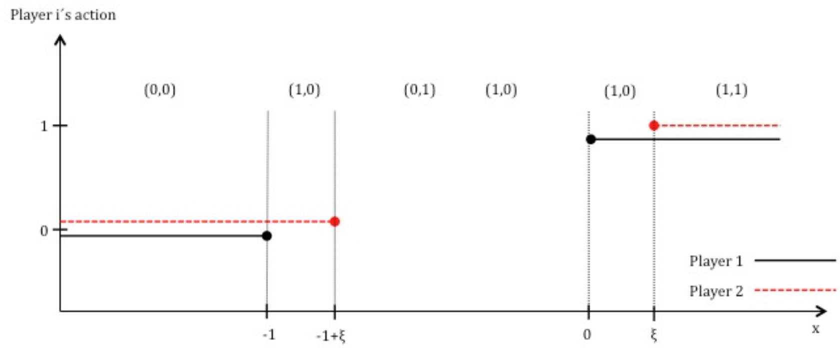

Figure 3: Nash equilibria sets with heterogenous players.

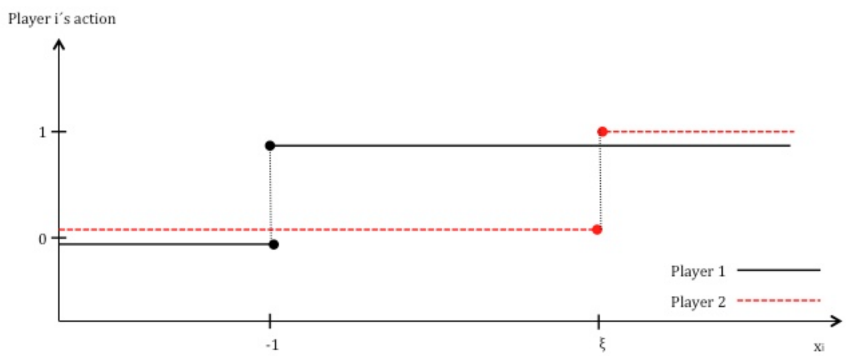

Figure 4: Unique equilibrium in the incomplete information game.

substitutes and many players. However, even though it is possible to do some strategy deletion, under his set up it is not possible to obtain the unique equilibrium through iterated elimination of strictly dominated strategies. This observation raises the natural question about the existence of sufficient conditions allowing to obtain this profile as the unique rationalizable outcome. We now present a simple three player global game with strategic substitutes where such conditions exists.

3 A simple global game with strategic substitutes

Dominance regions play a central role in the process of equilibrium selection in global games. In general, global games with strategic complements require the existence of these regions (plus other assumptions related with continuity and monotonicity of the payoff function) in order to do the equilibrium selection through iterative elimination of strategies. However, following Harrison (2003), we realize that, in the strategic substitute case, in order to start any process of elimination we need, additionally to the existence of dominance regions, some player' heterogeneity. Nevertheless, this condition is not sufficient for dominance solvability.

3.1 The Complete Information Game

Consider a three player two action game with strategic substitutes characterized by the following payoffs:[4]

$$u_i(a, x) = a_i \left(\frac{d}{2} (3 - a_1 - a_2 - a_3) + mx - c_i \right)$$

for $i \in \{1, 2, 3\}$ with $a_i \in \{0, 1\}$, $0 < c_1 < c_2 < c_3$ and $x \in \mathbb{R}$. Note that player i's payoff function is of the form

$$u_i(a, x) = \pi_i \left(a_i, \sum_{j \neq i} a_j, x \right)$$

Where $\pi_i : \{0, 1\} \times \{0, 1, 2\} \times \mathbb{R} \to \mathbb{R}$ is an auxiliary function that depends on other players' actions through their sum (the number of players (other than i) that are choosing action 1).

Let us define $\Delta\pi_i(n, x) = \pi_i(1, n, x) - \pi_i(0, n, x)$ as agent i's payoff difference when she is choosing action 1 rather than action 0. This is, the gain of player i of playing 1 instead of 0. Then

$$\Delta\pi_i(n, x) = \frac{d}{2} (2 - n) + mx - c_i$$

Which can be written as

$$\Delta\pi_i(n, x) = \Delta\pi(n, x) - c_i$$

where

$$\Delta\pi(n, x) := \frac{d}{2} (2 - n) + mx$$

Note that in this model

$$\begin{aligned} \Delta\pi_i(n, x) - \Delta\pi_i(n + 1, x) &= \Delta\pi(n, x) - c_i - (\Delta\pi(n + 1, x) - c_i) \\ &= \Delta\pi(n, x) - \Delta\pi(n + 1, x) \\ &= d \end{aligned} \tag{1}$$

so the parameter $d > 0$ represents the degree of strategic substitution. If $m > 0$ this payoff structure satisfies assumptions (A1) through (A5) of Harrison (2003).

Dominance regions

We can depict the dominance regions for each player as in Figure 5. The values $\underline{k}_i$ and $\bar{k}_i$ are defined by

$$\Delta\pi_i(0, \underline{k}_i) = 0 \qquad \qquad \Delta\pi_i\big(2, \bar{k}_i\big) = 0$$

and take the values

$$\underline{k}_i = \frac{c_i - d}{m} \qquad \qquad \bar{k}_i = \frac{c_i}{m}$$

If $x < \underline{k}_i$, then player i has as dominant action $a_i = 0$ and if $x > \bar{k}_i$ the dominant action of player i is $a_i = 1$.

[4]This game is inspired in the game presented in Morris and Shin (2009).

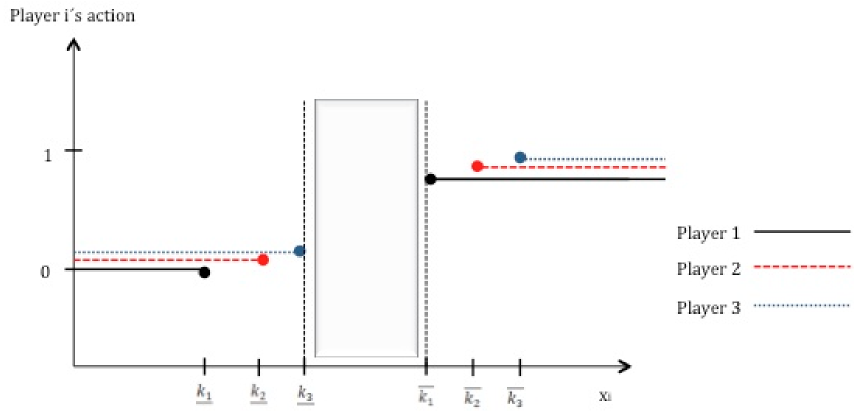

Figure 5: Dominance regions in three player two action global game with strategic substitutes.

To provide the unique equilibrium of the incomplete information game (see section 3.2) we need to define as well the value k_i as:

$$\Delta\pi_i(1, k_i) = 0,$$

which takes the value

$$k_i = \frac{2c_i - d}{2m}.$$

3.2 Incomplete Information

Consider now the three player incomplete information game $\Gamma(\sigma)$, consisting of the previous payoff structure and where each player has some uncertainty about x. Instead of observing the actual value of x, each player just observes a private signal x_i, which contains diffuse information about x, which is composed of the true value plus some noise:

$$x_i = x + \sigma\varepsilon_i$$

where $\sigma > 0$ is a scale factor, x is drawn from an interval $[\underline{X}, \overline{X}]$ with uniform density, and $\varepsilon_i \sim U$ with support in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In this context signals x_i belong to the set $X(\sigma) = [\underline{X} - \frac{1}{2}\sigma, \overline{X} + \frac{1}{2}\sigma]$.

If $\varepsilon_i - \varepsilon_j$ is distributed according to H, then $1 - F(x_i|x_j) = \Pr(x_i \geq t|x_j)$ may be calculated using H as follows:

$$\begin{aligned} 1 - F(x_i|x_j) &= \Pr(x_i \geq t|x_j) \\ &= \Pr\Big(\varepsilon_i - \varepsilon_j \geq \tfrac{(t-x_j)}{\sigma}\Big) \\ &= 1 - \Pr\Big(\varepsilon_i - \varepsilon_j < \tfrac{(t-x_j)}{\sigma}\Big) \\ &= 1 - H\Big(\tfrac{(t-x_j)}{\sigma}\Big). \end{aligned}$$

In this note we assume that H is the cdf function of the uniform distribution on $[-1,1]$. This is

$$H(t) = \begin{cases} 0 & \text{if } t < -1 \\ \frac{t+1}{2} & \text{if } -1 \leq t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}.$$

This general noise structure has been used in the global game literature, allowing the conditional distribution of the opponents signal to be modeled in a simple way, i.e. given a player's own signal, the conditional distribution of an opponent's signal x_j admits a continuous density f_σ and a cdf F_σ with support in the interval $[x_i - \sigma, x_i + \sigma]$. Moreover this literature establishes a significant result: when the prior is uniform, players' posterior beliefs about the difference between their own observation and other players' observations are the same,[5] i.e. $F_\sigma(x_i \mid x_j) = 1 - F_\sigma(x_j \mid x_i)$.

In this context of incomplete information, a Bayesian pure strategy for a player i is a function $s_i : X(\sigma) \to \{0,1\}$. A pure strategy profile is denoted as $s = (s_1, s_2, s_3)$ and $s_i \in S_i$, the set of all functions from $X(\sigma)$ to $\{0,1\}$.

A switching strategy of player i is a Bayesian pure strategy s_i satisfying:

$$\exists\ y \text{ s.t. } s_i(x_i) = \begin{cases} 0 & \text{if } x_i < y \\ 1 & \text{if } x_i > y \end{cases} \tag{2}$$

Abusing notation, we write $s_i^w(\cdot; y)$ to denote the switching strategy of player i with switching threshold y.

Consider the following strategy profile s^*:

$$s_1^* = s_1^w(\cdot; \underline{k}_1) \qquad s_2^* = s_2^w(\cdot; k_2) \qquad s_3^* = s_3^w(\cdot; \bar{k}_3)$$

Following Harrison (2003), if σ is sufficiently small the strategy profile s^* is the unique BNE of $\Gamma(\sigma)$. This results does not tell anything about dominance solvability. The proof Harrison (2003) utilizes a much stronger concept than iterative elimination of never best responses. Moreover, it is possible to give examples where the unique equilibrium is not strongly rational (Morris, 2009). In the next section we present a sufficient condition under which s^* is the only rationalizable outcome of $\Gamma(\sigma)$.

4 Main Result

Proposition 1. *If*

$$c_3 - c_1 > \frac{d}{2}$$

then $\exists\ \overline{\sigma}(c_1, c_2, c_3, d, m) > 0$ *such that* $\forall\ \sigma < \overline{\sigma}(c_1, c_2, c_3, d, m)$*, the set of rationalizable strategy profiles is equal to* $\text{BNE}(\Gamma(\sigma)) = \{s^*\}$.

Corollary 1. *If* $c_3 - c_1 > \frac{d}{2}$ *and* $\sigma \in\ ]0, \overline{\sigma}(c_1, c_2, c_3, d, m)[$*, then the unique equilibrium of* $\Gamma(\sigma)$ *is Strongly Rational.*

[5]This property holds approximately when x is not distributed with uniform density but σ is small, i.e. $F(x_i \mid x_j) \approx 1 - F(x_j \mid x_i)$ as σ goes to zero. See details in Lemma 4.1 Carlsson and van Damme (1993).

The proposition states that for a given degree of substitution (d) if players are sufficiently heterogenous $c_3 - c_1 > \frac{d}{2}$, or equivalently, if given players' heterogeneity the degree of strategic substitution is sufficiently small, then we get dominance solvability in $\Gamma(\sigma)$.

Proof. We show that the process of elimination of strictly dominated strategies delivers the unique equilibrium of the game. For this, we study a process of elimination of strategies that, in each step, finds as unreasonable for players to use one of the two actions when their signal is below or above certain values of their signals. These values are updated on each step and so constitute two sequences for each player i. The sequence $\{\bar{x}_i^t\}_{t=0}^{\infty}$ that decreases starting from $\bar{k}_i$, is such that at each step t player 3 has as dominant action the action 0 when her signal es above $\bar{x}_3^t$ and below $\bar{k}_3$ and player $i \in \{1, 2\}$ has as dominant action the action 1 when her signal es above $\bar{x}_i^t$. The sequence $\{\underline{x}_i^t\}_{t=0}^{\infty}$ that increases starting from $\underline{k}_i$, is such that at each step t player 1 has as dominant action the action 1 when her signal es below $\underline{x}_1^t$ and above $\underline{k}_1$ and player $i \in \{2, 3\}$ has as dominant action the action 0 when her signal es below $\underline{x}_i^t$. We show that under the assumptions of the Proposition, if σ is sufficiently small, then these sequences cross a threshold equal to the switching point of the equilibrium strategy of player 2, k_2, thus fixing the only possible remaining strategies at the equilibrium ones.

Since we have the dominance regions, we know that in any reasonable strategy player i plays 0 when the signal is below $\underline{k}_i$ and plays 1 if the signal is above $\bar{k}_i$. Thus, we start the process of elimination of strategy profiles by considering, for each player, strategies of the form:

$$s_i(x_i) = \begin{cases} 0 & \text{if } x_i < \underline{k}_i \\ 1 & \text{if } x_i > \bar{k}_i \end{cases}$$

Between $\underline{k}_i$ and $\bar{k}_i$ the strategies may take any value. We make the analysis for values $\underline{x}_i^t$ under which the strategies become fixed on each step t, the analysis for values $\bar{x}_i^t$ over which the strategies become fixed is analogous.

Set then $\underline{x}_i^0 = \underline{k}_i$. Since the game is of strategic substitutes, we can consider that players observe a worst case scenario and update the values $\underline{x}_i^t$ under which her strategies are fixed. Since want to isolate the equilibrium strategy profile, we want to show that for player 1, receiving a signal below $\underline{x}_1^t$ makes her play 1 (since below $\underline{k}_1$ her strategy is already fixed at 0) and for players 2 and 3 receiving a signal below $\underline{x}_i^t$ makes them play 0. If this is true for the worst case scenario, it will be true for any scenario.

We now state the update equations and take limit when $t \to \infty$ to find the limit of the sequence $\underline{x}^t$. The worst case scenario equations for each player are:

$$0 = \Delta\pi\big(0, \underline{x}_1^t\big)\, F\big(\underline{x}_2^{t-1} \mid \underline{x}_1^t\big)\, F\big(\underline{x}_3^{t-1} \mid \underline{x}_1^t\big) + \Delta\pi\big(1, \underline{x}_1^t\big) \Big[F\big(\underline{x}_2^{t-1} \mid \underline{x}_1^t\big)\, \big(1 - F\big(\underline{x}_3^{t-1} \mid \underline{x}_1^t\big)\big) + \\ + \big(1 - F\big(\underline{x}_2^{t-1} \mid \underline{x}_1^t\big)\big)\, F\big(\underline{x}_3^{t-1} \mid \underline{x}_1^t\big) \Big] + \Delta\pi\big(2, \underline{x}_1^t\big)\, \big(1 - F\big(\underline{x}_2^{t-1} \mid \underline{x}_1^t\big)\big)\, \big(1 - F\big(\underline{x}_3^{t-1} \mid \underline{x}_1^t\big)\big) - c_1; \tag{3}$$

$$0 = \Delta\pi\big(0, \underline{x}_2^t\big)\, \big(1 - F\big(\underline{x}_1^t \mid \underline{x}_2^t\big)\big) + \Delta\pi\big(1, \underline{x}_2^t\big)\, F\big(\underline{x}_1^t \mid \underline{x}_2^t\big) - c_2; \tag{4}$$

$$0 = \Delta\pi\big(0, \underline{x}_3^t\big)\, \big(1 - F\big(\underline{x}_1^t \mid \underline{x}_3^t\big)\big) + \Delta\pi\big(1, \underline{x}_3^t\big)\, F\big(\underline{x}_1^t \mid \underline{x}_3^t\big) - c_3. \tag{5}$$

The right hand side of these equations is the expected gain of playing action 1 under worst case scenarios (i.e. the lowest possible expected payoff), given that players know that the strategies of their opponentes are fixed for signal values smaller than $\underline{x}_i^t$. The updating occurs as follows: given the actual values of the sequences of players 2 and 3, we update the value of the sequence of player 1. With his new value of player 1, we update the values of players 2 and 3. We now study the limits of the sequences $\underline{x}_i^t$.

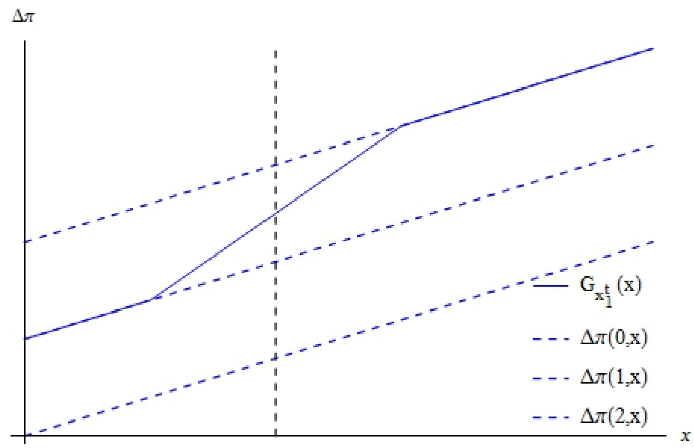

Figure 6: The function $G_{\underline{x}_1^t}(y)$. The vertical lines is $\underline{x}_1^t$.

Note that for equations (4) and (5), the right hand sides as functions of each player's signal, are equal to functions $G_{\underline{x}_1^t}(y) - c_i$, where $G_{\underline{x}_1^t}(y) := \Delta\pi(0,y)\,(1 - F(\underline{x}_1^t \mid y)) + \Delta\pi(1,y)\,F(\underline{x}_1^t \mid y)$ is depicted in Figure 6. When y is small, this function is equal to $\Delta\pi(1,y)$, then jumps continuously from $\Delta\pi(1,y)$ to $\Delta\pi(0,y)$ when y is in the σ neighborhood of $\underline{x}_1^t$ and then it becomes equal to $\Delta\pi(0,y)$. We may consider then the equation:

$$c = G_{\underline{x}_1^t}(y)\,.$$

Thus, given $\underline{x}_1^t$ the solution $\underline{x}_i^t$ of (4) and (5) (resp.) is either on the jump or beyond it (if it was before we would get as solutions $\underline{k}_i$ which can not be). If the solution is beyond the jump, we have gone all the way to $\bar{k}_i$ and thus the sequence would have collided with $\bar{x}_i^t$ and we would have already isolated the equilibrium strategy for player i, so we assume that on each t, the solution is in the jump and denote this solution as $Y(\underline{x}_1^t, c)$. Then

$$Y\left(\underline{x}_1^t, c\right) = \frac{d\underline{x}_1^t + 4c\sigma - 3d\sigma}{d + 4m\sigma}$$

and so

$$\begin{aligned} \underline{x}_2^t &= Y\left(\underline{x}_1^t, c_2\right) \\ &= \frac{d\underline{x}_1^t + 4c_2\sigma - 3d\sigma}{d + 4m\sigma} \qquad (6) \\ \underline{x}_3^t &= Y\left(\underline{x}_1^t, c_3\right) \\ &= \frac{d\underline{x}_1^t + 4c_3\sigma - 3d\sigma}{d + 4m\sigma}. \qquad (7) \end{aligned}$$

The right hand side of equation (3) can also be decomposed as a function of $y = \underline{x}_1^t$ minus c_1. The function is equal to $\Delta\pi(0, y)$ when y is small, then it jumps down (because σ is small) continuously to $\Delta\pi(2, y)$ when near $\underline{x}_2^{t-1}$ and $\underline{x}_3^{t-1}$ and then it is equal to $\Delta\pi(2, y)$ when y is large (see Figure 7). Equation (3) seeks a point such that this function is equal to c_1.

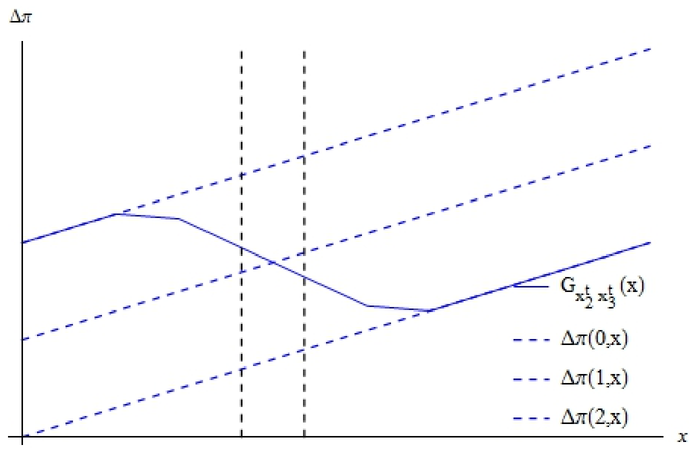

Figure 7: The expected payoff for player 1. The vertical lines are $\underline{x}_2^t$ and $\underline{x}_3^t$.

We have then either one or three solutions. If there is only one solution, then it is either $\underline{k}_1$ or $\bar{k}_1$. In the first case we must interpret that $\underline{x}_1^t$ is ∞ and we get a similar conclusion as above and the problem is solved. The second case is not possible under our assumptions and if we have three solutions then $\underline{x}_1^t$ is in the jump. In this case and since $\underline{x}_2^t$ and $\underline{x}_3^t$ are both in the jump of their function, we must have that at the jump both $F\left(\underline{x}_2^{t-1} \mid \underline{x}_1^t\right)$ and $F\left(\underline{x}_3^{t-1} \mid \underline{x}_1^t\right)$ are different from 0 and 1 and so we may replace them by the expression for the increasing part of F. This gives:

$$\underline{x}_1^t = \frac{d\left(\underline{x}_2^{t-1} + \underline{x}_3^{t-1} + 2\sigma\right) - 4c\sigma}{2\left(d - 2m\sigma\right)}. \tag{8}$$

Now, plugging (6) and (7) into (8) we get $\underline{x}_1^t$ as a function of $\underline{x}_1^{t-1}$.

$$\begin{aligned}\underline{x}_1^t &= \frac{d\left(\frac{d\underline{x}_1^{t-1}+4c_2\sigma-3d\sigma}{d+4m\sigma} + \frac{d\underline{x}_1^{t-1}+4c_3\sigma-3d\sigma}{d+4m\sigma} + 2\sigma\right) - 4c\sigma}{2\left(d - 2m\sigma\right)} \\ &= \frac{d^2\underline{x}_1^{t-1} + 2\left(c_3 - c_1 + c_2 - d\right)d\sigma + 4\left(d - 2c_1\right)m\sigma^2}{\left(d - 2m\sigma\right)\left(d + 4m\sigma\right)}.\end{aligned}$$

Taking the limit when $t \to \infty$ we get

$$\underline{x}_1^\infty = \frac{d^2\underline{x}_1^\infty + 2\left(c_3 - c_1 + c_2 - d\right)d\sigma + 4\left(d - 2c_1\right)m\sigma^2}{\left(d - 2m\sigma\right)\left(d + 4m\sigma\right)}.$$

which gives

$$\underline{x}_1^\infty = \frac{\left(c_3 - c_1 + c_2 - d\right)d + 2\left(d - 2c_1\right)m\sigma}{m\left(d - 4m\sigma\right)}$$

We can now calculate $\underline{x}_2^\infty$ and $\underline{x}_3^\infty$:

$$\underline{x}_2^\infty = Y(\underline{x}_1^\infty, c_2)$$
$$\underline{x}_3^\infty = Y(\underline{x}_1^\infty, c_3)$$

If we take $\sigma \to 0$ we obtain that for $i \in \{1, 2, 3\}$, $\underline{x}_i^\infty \to \frac{c_3 - c_1 + c_2 - d}{m}$ and if $c_3 - c_1 > \frac{d}{2}$, we get that

$$\lim_{\sigma \to 0} \underline{x}_2^\infty = \frac{c_3 - c_1 + c_2 - d}{m} > \frac{2c2 - d}{2m} = k_2.$$

So given (c_1, c_2, c_3, d, m), there exists a threshold σ_b, that depends on (c_1, c_2, c_3, d, m), such that if σ is smaller than σ_b, then the sequence starting from below for player 2 converges to the right of k_2.

By the analogous exercise developed from above,[6] we will get that given (c_1, c_2, c_3, d, m), there exists a threshold σ_a, that depends on (c_1, c_2, c_3, d, m), such that if σ is smaller than σ_a, then the sequence starting from above for player 2 converges to the left of k_2.

So if σ is smaller than the threshold for Harrison's uniqueness theorem and smaller than $\min\{\sigma_a, \sigma_a\}$, then the only strategy of player 2 isolated by the process of iterated elimination is her unique equilibrium strategy, implying that for all three players the only strategies isolated by the process of iterated elimination are the unique equilibrium strategies.

□

5 Conclusions

We have presented a simple model of a global game with strategic substitutes for which we provide a sufficient condition for strong rationality of its unique equilibrium.

The condition states for this class of games that a sufficient negative correlation between the degree of strategic substitution and heterogeneity among players, allows dominance solvability. This can be achieved by a minimum players' heterogeneity for a given degree of strategic substitution or equivalently, by a maximal degree of strategic substitution for a given players' heterogeneity.

The intuition behind this condition for dominance solvability relies on the structure of the process of iterated elimination of strictly dominated strategies. It is not only necessary that the lower cost the player is, the more the incentive she has to have to pick the higher action ($a = 1$), but also some additional requirements are needed such that allow players to form beliefs consistent with a process of elimination that reaches to a single profile. In this sense, it will be required that, for a lower cost player to pick the higher action, we need the existence of a minimum of players' heterogeneity or a maximum degree of strategic substitution, because it is the only way that she can believe that the probabilities that "higher cost (than her) players" are picking the higher action are very low.

Our result is consistent with Guesnerie and Jara-Moroni (2011) and Morris and Shin (2009) and suggests that there may exist strongly rational equilibrium in a more general set up of global games with strategic substitutes. From the condition on Proposition 1 and equation (1) we infer that the result may be generalized for more general payoffs of the form

[6] The details are available from the authors upon request.

$\Delta\pi(n, x) - c_i$ that satisfy assumptions (A1) to (A5) in Harrison (2003). This last remark opens an unexplored research agenda on the study of strongly rational equilibria in global games with strategic substitutes.

References

Carlsson, H. and E. van Damme (1993, September). Global games and equilibrium selection. *Econometrica 61*(5), 989–1018.

Chamley, C. (1999, August). Coordinating regime switches. *The Quarterly Journal of Economics 114*(3), 869–905.

Chamley, C. P. (2004). *Rational Herds: Economic Models of Social Learning*. Cambridge University Press.

Desgranges, G. and M. Heinemann (2005, March). Strongly rational expectations equilibria with endogenous acquisition of information. Working Paper Series in Economics 9, University of Lüneburg, Institute of Economics.

Evans, G. W. and R. Guesnerie (1993). Rationalizability, strong rationality and expectational stability. *Games and Economic Behavior 5*, 632–646.

Evans, G. W. and R. Guesnerie (2003). Coordination on saddle-path solutions: The eductive viewpoint–linear univariate models. *Macroeconomic Dynamics 7*(1), 42–62.

Evans, G. W. and R. Guesnerie (2005, October). Coordination on saddle-path solutions: the eductive viewpoint–linear multivariate models. *Journal of Economic Theory 124*(2), 202–229.

Frankel, D. M., S. Morris, and A. Pauzner (2003, January). Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory 108*(1), 1–44.

Guesnerie, R. (1992, December). An exploration on the eductive justifications of the rational-expectations hypothesis. *The American Economic Review 82*(5), 1254–1278.

Guesnerie, R. (2002, March). Anchoring economic predictions in common knowledge. *Econometrica 70*(2), 439.

Guesnerie, R. and P. Jara-Moroni (2011). Expectational coordination in simple economic contexts. *Economic Theory 47*, 205–246. 10.1007/s00199-010-0556-8.

Harrison, R. (2003). Global games with strategies substitutes. Available at http://www.economia.puc.cl/archivos_profes/12/GlobalGames_Harrison.pdf.

Morris, S. (2009). Three player global game with strategic substitutes and many rationalizable actions. Personal comunication.

Morris, S. and H. S. Shin (1998, June). Unique equilibrium in a model of self-fulfilling currency attacks. *American Economic Review 88*(3), 587–597.

Morris, S. and H. S. Shin (2003). Global games: Theory and applications. In M. Dewatripont, L. Hansen, and S. Turnovsky (Eds.), *Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress Vol I*, Volume 1, pp. 57–114. Cambridge University Press.

Morris, S. and H. S. Shin (2005, January). Notes on strategic substitutes and complements in global games. Available at web page.

Morris, S. and H. S. Shin (2009). Coordinating expectations: Global games with strategies substitutes.